An Efficient Algorithm for Computing High-Quality Paths amid Polygonal Obstacles*

Pankaj K. Agarwal¹, Kyle Fox¹, and Oren Salzman²

¹Duke University, Durham, North Carolina, USA, {pankaj,kylefox}@cs.duke.edu
²Tel Aviv University, Tel Aviv, Israel, orenzelz@post.tau.ac.il

Abstract
We study a path-planning problem amid a set $\Omega$ of obstacles in $\mathbb{R}^2$, in which we wish to compute a short path between two points while also maintaining a high clearance from $\Omega$; the clearance of a point is its distance from a nearest obstacle in $\Omega$. Specifically, the problem asks for a path minimizing the reciprocal of the clearance integrated over the length of the path. We present the first polynomial-time approximation scheme for this problem. Let $n$ be the total number of obstacle vertices and let $\varepsilon \in (0, 1]$. Our algorithm computes in time $O(n^2 \log \frac{n}{\varepsilon})$ a path of total cost at most $(1+\varepsilon)$ times the cost of the optimal path.

1 Introduction

Motivation. Robot motion planning deals with planning a collision-free path for a moving creature in an environment cluttered with obstacles [6]. It has applications in diverse domains such as surgical planning and computational biology. Typically, a high-quality path is desired where quality can be measured in terms of path length, clearance (distance from nearest obstacle at any given time), or smoothness, to mention a few criteria.

Problem statement. Let $\Omega$ be a set of polygonal obstacles in the plane, consisting of $n$ vertices in total. A path $\gamma$ for a point robot moving in the plane is the image of a continuous function $\gamma : [0, 1] \rightarrow \mathbb{R}^2$. Let $||p,q||$ denote the Euclidean distance between two points $p,q$. The clearance of a point $p$, denoted by $\text{cl}(p) := \min_{o \in \Omega} ||p,o||$, is the minimal Euclidean distance between $p$ and an obstacle ($\text{cl}(p) = 0$ if $p$ lies in an obstacle). Similarly, the clearance of a path $\gamma$ is defined as $\text{cl}(\gamma) := \min_{\tau \in [0,1]} \text{cl}(\gamma(\tau))$. We use the following cost function, as defined by Wein et al. [17], that takes both the length and the clearance of a path $\gamma$ into account:

$$\mu(\gamma) := \int_0^1 \frac{1}{\text{cl}(\gamma(\tau))} d\tau. \quad (1.1)$$

This criteria is useful in many situations because we wish to find a short path that does not pass too close to the obstacles due to of safety requirements. We abuse notation and let $\mu(p,q)$ be the minimal cost of any path between $p$ and $q$.

The (approximate) minimal-cost path problem is defined as follows: Given the set of obstacles $\Omega$ in $\mathbb{R}^2$, a real number $\varepsilon \in (0, 1]$, a start position $s$ and a target position $t$, compute a path between $s$ and $t$ with cost at most $(1+\varepsilon) \cdot \mu(s,t)$.

Related work. There is extensive work in computational geometry on computing shortest collision-free paths for a point moving amid a set of planar obstacles, and by now optimal $O(n \log n)$ algorithms are known; see Mitchell [12] for a survey and [5, 10] for recent results. There is also work on computing paths with the minimum number of links [13]. A drawback of these paths is that they may touch obstacle boundaries and therefore their clearance may be zero. Conversely, if maximizing the distance from the obstacles is the optimization criteria, then the path can be computed by constructing a maximum spanning tree in the Voronoi diagram of the obstacles (see O’Dunlaing and Yap [14]). Wein et al. [16] considered the problem of computing shortest paths that have clearance at least $\delta$ for some parameter $\delta$. However, this measure does not quantify the trade-off between the length and the clearance, so Wein et al. [17] suggested the cost function defined in (1.1)

---

*Work by P.A. and K.F. has been supported in part by NSF under grants CCF-09-40671, CCF-10-12254, CCF-11-61359, IIS-14-08846, and CCF-15-13816, and by Grant 2012/229 from the U.S.-Israel Binational Science Foundation. Work by O.S. has been supported in part by the Israel Science Foundation (grant no.1102/11), by the German-Israeli Foundation (grant no. 1150-82.6/2011), and by the Hermann Minkowski–Minerva Center for Geometry at Tel Aviv University.

**Wein et al. assume the minimal-cost path exists. One can formally prove its existence by taking the limit of paths approaching the infimum cost.**
to balance between minimizing the path length while maximizing its clearance. They devise an approximation algorithm to compute near-optimal paths under this metric for a point robot moving amidst polygonal obstacles in the plane. Their approximation algorithm runs in time polynomial in $\frac{1}{\epsilon}, n$ and $\Lambda$ where $\epsilon$ is the maximal additive error, $n$ is the number of obstacle vertices and $\Lambda$ is (roughly speaking) the total cost of the edges in the Voronoi diagram of the obstacles\(^2\). We are not aware of any polynomial-time approximation algorithm for this problem. It is not known whether the problem of computing the optimal path is NP-hard.

The problem of computing shortest paths amid polyhedral obstacles in $\mathbb{R}^3$ is NP-Hard [3], and a few heuristics have been proposed in the context of sampling-based motion planning in high dimensions (a widely used approach in practice [6]) to compute a short path that has some clearance; see, e.g., [15].

Several other bicriteria measures have been proposed in the context of path planning amid obstacles in $\mathbb{R}^2$, which combine the length of the path with curvature, the number of links in the path, the visibility of the path, etc. (see e.g. [1,4,11] and references therein). We also note a recent work by Cohen et al., which is dual to the problem studied here [7]: Given a point set $P$ and a path $\gamma$, they define the cost of $\gamma$ to be the integral of clearance along the path, and the goal is to compute a minimal-cost path between two given points. They present an approximation algorithm whose running time is near-linear in the number of points.

**Our Contribution.** We present an algorithm\(^3\) that given $O, s, t$ and $\epsilon \in (0,1)$, computes in time $O\left(\frac{n^2}{\epsilon^2} \log \frac{1}{\epsilon}\right)$ a path from $s$ to $t$ whose cost is at most $(1 + \epsilon) \cdot \mu(s, t)$.

As in [17], our algorithm is based on sampling, i.e., it contains a weighted geometric graph $G = (V,E)$ with $V \subset \mathbb{R}^2$ and $s,t \in V$ and computes a minimal-cost path in $G$ from $s$ to $t$. However, we prove a number of useful properties of optimal paths to obtain a fast algorithm.

We first refine the Voronoi diagram $V$ of $O$ into constant-size cells, which we refer to as the refined Voronoi diagram of $O$ and denote it by $\overline{V}$. We prove (in Section 3) the existence of a path $\gamma$ from $s$ to $t$ whose cost is $O(\mu(s,t))$ and that has the following useful properties: (i) for every cell $T \in \overline{V}$, $\gamma \cap \text{int}(T)$ is a connected subpath and the clearance of all points in this subpath is the same; (ii) for every edge $e \in \overline{V}$, there are $O(1)$ points, called anchor points, that depend only on the two cells incident to $e$ with the property that either $\gamma$ intersects $e$ transversally (i.e., $\gamma \cap e$ is a single point) or the endpoints of the closure of $\gamma$ intersect $e$ at anchor points. We say $\gamma$ consists of well-behaved paths. We use anchor points to propose a simple $O(n)$-approximation algorithm (Section 4.1), which we then transform into an $O(1)$-approximation algorithm (Section 4.2). We also use anchor points and the existence of well-behaved paths to choose a set of $O(n)$ or $O(n/\epsilon)$ sample points on each edge of $\overline{V}$ with a total of $O(n^2)$ or $O(n^2/\epsilon)$ samples in total (Sections 4.2 and 4.3).

We prove additional properties of optimal paths to construct the final graph with $O((n^2/\epsilon^2) \log n/\epsilon)$ edges (Section 4.3) instead of connecting every pair of sample points by an edge. Roughly speaking, we show that one can construct a small-size spanner.

2 Preliminaries

Recall that $O$ is a set of polygonal obstacles in the plane consisting of $n$ vertices in total. We refer to the edges and vertices of $O$ as its features. Given a point $p$ and a feature $o$, let $\psi_o(p)$ be the closest point to $p$ on $o$ so that $\|p, o\| = \|p, \psi_o(p)\|$. If a path $\gamma$ contains two points $p$ and $q$, we let $\gamma[p,q]$ denote the subpath of $\gamma$ between $p$ and $q$.

**Voronoi diagram and its refinement.** The Voronoi cell of a polygon feature $o$ is the set of points in the closure of $\mathbb{R}^2 \setminus (\bigcup O)$ whose distance to any feature in $O$ is minimized by $o$. The Voronoi cells of features are connected and internally disjoint. The Voronoi diagram $V$ of features in $O$ is the planar subdivision of the closure of $\mathbb{R}^2 \setminus (\bigcup O)$ determined by the Voronoi cells of features in $O$. Voronoi diagram edges between a line and a point obstacle’s cells are parabolic arcs, and Voronoi edges between two line obstacles’ or two point obstacles’ cells are line segments. The Voronoi diagram has total complexity $O(n)$. See [2] for details.

Note that for any obstacle feature $o$, and for any point $x$ along any Voronoi edge on the boundary of $o$’s Voronoi cell the function $||x, \psi_o(x)||$ is convex. We define the refined Voronoi diagram $\overline{V}$ by adding the following edges to $V$: (i) the line segments $x\psi_o(x)$ between each obstacle feature $o$ and Voronoi vertex $x$ on the boundary of $o$’s Voronoi cell, (ii) the line segment $x\psi_o(x)$ between each obstacle feature $o$ and the point $x$ along each Voronoi edge bounding $o$’s cell that minimizes $||x, \psi_o(x)||$ , and (iii) a line segment from the obstacle feature $o$ closest to $s$ (to $t$) that initially follows $\phi_o(s)$ (or $\phi_o(t)$) and ends at the first Voronoi edge it intersects. We refer to these extra edges as type (i), type (ii), or type (iii) edges respectively. Note that some type (i) edges may

---

\(^2\)For the exact definition of $\Lambda$, see [17].

\(^3\)We assume in this paper that $\mathbb{R}^2 \setminus (\bigcup O)$ is bounded. Our algorithm can be modified easily to avoid this assumption.
already be present in the Voronoi diagram $\mathcal{V}$. We say that an edge in $\mathcal{V}$ is an *internal edge* if it separates two cells incident to the same polygon. Other edges are called *external edges*.

Clearly, the complexity of $\hat{\mathcal{V}}$ is $O(n)$. Moreover, each cell $T$ in $\hat{\mathcal{V}}$ is incident to a single obstacle feature and has three additional edges. One edge (an external edge) of $T$ is a monotone parabolic arc (we view a line segment as a parabolic arc); it is incident to two internal edges on $T$. For each cell $T$, let $\kappa_T$ be the external edge of $T$, let $\alpha_T$ and $\beta_T$ be the shorter and longer internal edges of $T$, respectively, and let $u_T$ and $v_T$ be the vertices connecting $\alpha_T$ and $\beta_T$ to $\kappa_T$ respectively. See Figure 1b.

**Properties of optimal paths.** We list several properties of our cost function. For detailed explanations and proofs, the reader is referred to Wein et al. [17]. Let $s = r_se^{i\theta_s}$ be a start position and $t = r_te^{i\theta_t}$ be a target position.

- Let $o$ be a point obstacle with $O = \{o\}$, and assume without loss of generality that $o$ lies at the origin and $0 \leq \theta_s \leq \theta_t \leq \pi$. The optimal path between $s$ and $t$ (see Figure 2a) is a logarithmic spiral centered on $o$, and its cost is

$$\mu(s,t) = \sqrt{(\theta_t - \theta_s)^2 + (\ln r_t - \ln r_s)^2}. \quad (2.2)$$

- Let $o$ be a line obstacle with $O = \{o\}$, and assume without loss of generality that $o$ is supported by the line $y = 0$ and $0 \leq \theta_s \leq \theta_t \leq \pi$. The optimal path between $s$ and $t$ (see Figure 2b) is a circular arc with its center at the origin, and its cost is

$$\mu(s,t) = \ln \frac{1 - \cos \theta_t}{\sin \theta_t} - \ln \frac{1 - \cos \theta_s}{\sin \theta_s} = \ln \frac{\sin \theta_t}{\theta_t} - \ln \frac{\sin \theta_s}{\theta_s}. \quad (2.3)$$

- Let $o$ be an obstacle with $O = \{o\}$ and $s$ on the line segment between $\psi_o(t)$ and $t$. The optimal path between $s$ and $t$ (see Figure 2c) is a line segment, and its cost is

$$\mu(s,t) = \ln c(t) - \ln c(s). \quad (2.4)$$

- The minimal-cost path $\gamma$ between two points $p$ and $q$ on an edge $e$ of $\mathcal{V}$ is the piece of $e$ between $p$ and $q$. Moreover, there is a closed-form formula describing the cost of $\gamma$.

- Since each point within a single Voronoi cell is closest to exactly one obstacle feature, we may conclude the following: Given a set of obstacles, the optimal path connecting $s$ and $t$ consists of a sequence of circular arcs, pieces of logarithmic spirals, line segments, and pieces of Voronoi edges (see Figure 2d).

The following Corollary follows immediately from (1.1) and (2.2).

4The original equation describing the cost of the optimal path in the vicinity of a line obstacle had the obstacle on $x = 0$ and contained a minor inaccuracy. We present the correct cost in (2.3).
(ii) Given a single point obstacle on the minimal-cost path from \( p, q \) between \( (p, q) \), then the bound is tight.

Model of computation. We are primarily concerned with the combinatorial time complexity of our algorithm. Therefore, we assume a model of computation that allows us to evaluate basic trigonometric and algebraic expressions in constant time such as the ones given above. Our model also allows us to find the roots of low degree polynomials in constant time.

3 Well-behaved Paths

Let \( T \) be a cell of \( \mathcal{V} \) incident to obstacle feature \( o \), and let \( p \) and \( q \) be two points on the edges of \( T \). Let \( \gamma \) be any path from \( p \) to \( q \) that does not leave \( T \). We say \( \gamma \) is a well-behaved path if (i) \( \gamma \cap \text{int}(T) \) is a connected subpath and (ii) if it exists, then \( \gamma \cap \text{int}(T) \) has constant clearance. For a well-behaved path \( \gamma \), let \( \lambda(\gamma) = \gamma \cap \text{int}(T) \). We often use \( \lambda \) in place of \( \lambda(\gamma) \) when \( \gamma \) is clear from the context. If \( o \) is a vertex, then \( \lambda \) is a circular arc centered at \( o \), and if \( o \) is an edge, then \( \lambda \) is a line segment parallel to \( o \). We have the following lemma.

Lemma 3.1. Let \( T \) be a cell of \( \mathcal{V} \), and let \( p \) and \( q \) be two points on the edges of \( T \). There exists a well-behaved \( (p, q) \)-path \( \gamma \) within \( T \) where \( \mu(\gamma) \leq 7\mu(p, q) \).

Proof. Let \( o \) be the obstacle feature incident to \( T \). Let \( cl_{\text{max}}(p, q) \) be the maximum clearance achieved by the minimal-cost path between \( p \) and \( q \). Let \( T' \) be the subset of \( T \) restricted to points of clearance at most \( cl_{\text{max}}(p, q) \). Path \( \gamma \) follows the unique path from \( p \) to \( q \) along the boundary of \( T' \) that does not intersect \( o \). In particular, if both \( p \) and \( q \) lie on the same edge of \( T \), then \( \gamma \) is the minimal-cost path from \( p \) to \( q \) (Corollary 2.1). We have three more cases to consider (and their symmetries).

Case 1) Points \( p \) and \( q \) lie on \( \alpha_T \) and \( \kappa_T \) respectively. Path \( \gamma \) travels along \( \alpha_T \) from \( p \) to \( w_T \), and then along \( \kappa_T \) from \( w_T \) to \( q \). By Corollary 2.1 and the fact that \( w_T \) is the lowest clearance point on \( \kappa_T \) relative to \( p \), we have \( \mu(p, w_T) \leq \mu(p, q) \). By the triangle inequality we have that \( \mu(w_T, q) \leq \mu(w_T, p) + \mu(p, q) \leq 2\mu(p, q) \). Finally, \( \mu(\gamma) \leq \mu(p, w_T) + \mu(w_T, q) \leq 3\mu(p, q) \). See Figure 3a.

Case 2) Points \( p \) and \( q \) lie on \( \beta_T \) and \( \kappa_T \) respectively. Let \( w \) be the endpoint of \( \lambda = \lambda(\gamma) \) on \( \beta_T \), and let \( w' \) be the endpoint lying on \( \kappa_T \). Path \( \gamma \) travels along \( \beta_T \) from \( p \) to \( w \), along \( \lambda \), and then along \( \kappa_T \) from \( w' \) to \( q \).

Again, \( \mu(p, w) \leq \mu(p, q) \). If the obstacle defining \( T \) is a polygon edge, then \( \lambda \) is the Euclidean shortest path between any pair of points on \( \beta_T \) and \( \kappa_T \) whose clearance never exceeds \( cl_{\text{max}}(p, q) \). It also (trivially) has the highest clearance of any such path. If the obstacle is a polygon vertex, then \( \lambda \) spans a shorter angle relative to the vertex than any other path whose clearance never exceeds \( cl_{\text{max}}(p, q) \). By Corollary 2.1, the cost of any path from \( \beta_T \) to \( \kappa_T \) is at least this angle, and by (2.2), the cost of \( \lambda \) is exactly this lower bound. Either way, any path between \( p \) and \( q \) also goes between \( \beta_T \) and \( \kappa_T \), so we conclude that \( \mu(\lambda) \leq \mu(p, q) \). We have \( \mu(w', q) \leq \mu(\lambda) + \mu(w, p) + \mu(p, q) \leq 3\mu(p, q) \). Therefore, \( \mu(\gamma) \leq \mu(p, w) + \mu(\lambda) + \mu(w', q) \leq 5\mu(p, q) \). See Figure 3b.

Case 3) Points \( p \) and \( q \) lie on \( \beta_T \) and \( \alpha_T \) respectively. Let \( w \) be the endpoint of \( \lambda = \lambda(\gamma) \) on \( \beta_T \) and let \( w' \) be the other endpoint. As before, \( \mu(p, w) \leq \mu(p, q) \)
and \( \mu(\lambda) \leq \mu(p, q) \).

Suppose \( w' \) is on \( \kappa_T \). Path \( \gamma \) travels along \( \beta_T \) from \( p \) to \( w \), \( \alpha_T \) to \( w' \), and then along \( \alpha_T \) to \( q \). We have \( c_l(p, q) \geq c_l(w') \geq c_l(w_T) \), so \( \mu(q, w_T) \leq \mu(p, q) \). Therefore \( \mu(w', w_T) \leq \mu(\lambda) + \mu(w, p) + \mu(p, q) + \mu(q, w_T) \leq 4\mu(p, q) \). Finally \( \mu(\gamma) = \mu(p, w) + \lambda + \mu(w', w_T) + \mu(w_T, q) \leq 7\mu(p, q) \). See Figure 3d.

Now, suppose \( w' \) is on \( \alpha_T \). Path \( \gamma \) travels along \( \beta_T \) from \( p \) to \( w \), \( \alpha_T \) and then along \( \alpha_T \) from \( w' \) to \( q \). We have \( \mu(q, w') \leq \mu(p, q) \). Therefore, \( \mu(\gamma) \leq \mu(p, w) + \lambda + \mu(w', q) \leq 3\mu(p, q) \). See Figure 3d.

In the proof of Lemma 3.1, we chose subpath \( \lambda = \lambda(\gamma) \) based on the maximum clearance of the minimal-cost \( (p, q) \)-path. Given \( \lambda \) and a point \( p \) on \( \beta_T \), let \( \gamma(p, \lambda) \) be the path that walks along \( \beta_T \) from \( p \) to \( \lambda \) and then walks along \( \lambda \). We argued that \( \mu(\gamma(p, \lambda)) \leq O(\mu(p, q)) \). Point \( w \) on \( \beta_T \) was the endpoint of \( \lambda \). In the following lemma, we prove the existence of two anchor points \( w^*_p \) and \( w^*_q \) on \( \beta_T \) which help us pick a suitable \( \lambda \) without knowing anything about the minimal-cost \( (p, q) \)-path other than its endpoint \( p \) (note that \( \lambda \) does not exist when neither \( p \) nor \( q \) use edge \( \beta_T \)). As we show, the anchor points can be computed in constant time given \( T \).

**Lemma 3.2.** Let \( T \) be a cell of \( \mathcal{V} \). There exist points \( w^*_p \) and \( w^*_q \) on \( \beta_T \) such that the following holds: Let \( p \) and \( q \) be two points on the edges of \( T \). There exists a well-behaved \( (p, q) \)-path \( \gamma \) within \( T \) such that \( \mu(\gamma) \leq 7\mu(p, q) \). If neither \( p \) nor \( q \) lie on \( \beta_T \), then \( \gamma \) stays on \( \alpha_T \) and \( \kappa_T \). Otherwise \( \lambda(\gamma) \cap \beta_T \in \{w^*_p, w^*_q, p, q\} \).

**Proof.** Let \( o \) be the obstacle feature incident to \( T \). We assume at least one of \( p \) and \( q \) lie on \( \beta_T \). Otherwise, we simply use the well-behaved path that follows one or both of \( \alpha_T \) and \( \kappa_T \). See the proof of Lemma 3.1.

Without loss of generality, \( p \) lies on \( \beta_T \). We will pick \( \lambda = \lambda(\gamma) \) so that we minimize the cost of \( \gamma(p, \lambda) \).

The lemma follows then by using our choice of \( \lambda \) in the proof of Lemma 3.1. Observe that for \( \lambda \) that minimizes the cost of \( \gamma(p, \lambda) \), the endpoint of \( \lambda \) on \( \beta_T \) never lies closer to \( o \) than \( p \); the value \( \mu(\lambda) \) cannot decrease as one moves \( \lambda \) below \( p \). We will now consider several cases based on the shape of \( T \)'s edges and the edge containing \( q \).

**Case 1** Suppose \( o \) is a vertex. Without loss of generality, \( o \) lies at the origin, edges \( \alpha_T \) and \( \beta_T \) intersect the line \( y = 0 \) at the origin with angles \( \theta_{\alpha_T} \) and \( \theta_{\beta_T} \) respectively, and \( \theta_{\beta_T} > \theta_{\alpha_T} \geq 0 \).

**Case 1a** Suppose \( \kappa_T \) is a line segment, and suppose \( q \) lies on \( \kappa_T \). Without loss of generality, \( \kappa_T \) is supported by the line \( x = r_T \). The equation of the line in polar coordinates is \( r = r_T / \cos \theta \). We have \( \theta_{\beta_T} \leq \pi / 2 \).

Let \( \theta \) be any value such that \( \theta_{\alpha_T} \leq \theta \leq \theta_{\beta_T} \). Suppose we choose \( \lambda \) such that \( \lambda \)'s endpoint on \( \kappa_T \) lies at angle \( \theta \) relative to the \( x \)-axis. As mentioned, the optimal choice for \( \theta \) guarantees \( c_l(\lambda) \geq c_l(p) \). We say \( \theta \) is feasible if \( c_l(\lambda) \geq c_l(p) \) and \( \theta_{\alpha_T} \leq \theta \leq \theta_{\beta_T} \) (see Figure 4a).

Recall Corollary 2.1. Restricting ourselves to feasible values of \( \theta \), we have

\[
\mu(\gamma(p, \lambda)) = \frac{c_l(\lambda)}{c_l(p)} + \frac{\theta_{\beta_T} - \theta}{\cos \theta}
\]

Taking the derivative, we see

\[
\frac{d}{d\theta} \mu(\gamma(p, \lambda)) = \tan \theta - 1.
\]

This expression is negative for \( \theta = 0 \), positive near \( \theta = \pi / 2 \), and it has at most one root within feasible values of \( \theta \), namely at \( \theta = \pi / 4 \). Therefore, \( \mu(\gamma(p, \lambda)) \) is minimized when either \( c_l(\lambda) = c_l(p) \) or \( \theta = \theta^* = \min(\pi / 4, \theta_{\alpha_T}, \theta_{\beta_T}) \). We pick \( w^*_\kappa \) so that \( c_l(w^*_\kappa) = r_T / \cos \theta^* \).

**Case 1b** Suppose \( \kappa_T \) is a parabolic arc, and suppose \( q \) lies on \( \kappa_T \). Without loss of generality,

![Figure 3: Different cases considered in the proof of Lemma 3.1 for a line-segment obstacle.](image-url)
the parabola supporting \( \kappa_T \) is equidistant between \( o \) and the line \( x = 2r_T \). The equation of the parabola in polar coordinates is \( r = 2r_T/(1 + \cos \theta) \). We have \( \theta_{\beta_T} \leq \pi \). Define \( \theta \) and choose \( \lambda \) as in Case 1a. Again, angle \( \theta \) is feasible if \( \text{cl}(\lambda) \geq \text{cl}(p) \) and \( \theta_{\alpha_T} \leq \theta \leq \theta_{\beta_T} \). These cases are handled above.

Recall Corollary 2.1. Restricting ourselves to feasible values of \( \theta \), we have

\[
\mu(\gamma(p, \lambda)) = \ln \frac{\text{cl}(\lambda)}{\text{cl}(p)} + \theta_{\beta_T} - \theta
\]

We have

\[
\frac{d}{d\theta} \mu(\gamma(p, \lambda)) = -\frac{\sin \theta}{1 + \cos \theta} - 1
\]

Again, the expression is negative for \( \theta = 0 \), positive for \( \theta \) near \( \pi \), and it has at most one root within feasible values of \( \theta \), namely at \( \theta = \pi/2 \). Therefore, \( \mu(\gamma(p, \lambda)) \) is minimized when either \( \text{cl}(\lambda) = \text{cl}(p) \) or \( \theta = \theta^* = \min\{\pi/2, \theta_{\alpha_T}, \theta_{\beta_T}\} \). We pick \( w^*_\kappa \) so that \( \text{cl}(w^*_\kappa) = 2r_T/(1 + \cos(\theta^*)) \).

**Case 1c** Suppose \( q \) lies on \( \alpha_T \). In some cases, such as when \( \text{cl}(p) > \text{cl}(w_T) \), it is best for \( \lambda \) to intersect \( \kappa_T \). These cases are handled above. Suppose it is better for \( \lambda \) to intersect \( \alpha_T \), and consider any such \( \lambda \). By Corollary 2.1, the cost of \( \lambda \) is simply \( \theta_{\beta_T} - \theta_{\alpha_T} \). Therefore, \( \mu(\gamma(p, \lambda)) \) is minimized when \( \text{cl}(\lambda) = \text{cl}(p) \). We (arbitrarily) pick \( w^*_\alpha \) so that \( \text{cl}(w^*_\alpha) = \text{cl}(w_T) \).

**Case 2** Suppose \( o \) is a polygon edge. Without loss of generality, \( o \) lies on the line \( y = 0 \), the edge \( \alpha_T \) lies on the line \( x = x_{\alpha_T} \), the edge \( \beta_T \) lies on the line \( x = x_{\beta_T} \), and \( x_{\beta_T} > x_{\alpha_T} \geq 0 \).

**Case 2a** Suppose \( \kappa_T \) is a line segment, and suppose \( q \) lies on \( \kappa_T \). Without loss of generality, the line supporting \( \kappa_T \) intersects \( o \) at the origin with angle \( \theta_\kappa \). Let \( x \) be any value such that \( x_{\alpha_T} \leq x \leq x_{\beta_T} \). Suppose we choose \( \lambda \) such that \( \lambda \)'s endpoint on \( \kappa_T \) has \( x \)-coordinate \( x \). As mentioned, the optimal choice for \( x \) guarantees \( \text{cl}(\lambda) \geq \text{cl}(p) \). We say \( x \) is feasible if \( \text{cl}(\lambda) \geq \text{cl}(p) \) and \( x_{\alpha_T} \leq x \leq x_{\beta_T} \) (see Figure 4b).

Recall Corollary 2.1. Restricting ourselves to feasible values of \( x \), we have

\[
\mu(\gamma(p, \lambda)) = \ln \frac{\text{cl}(\lambda)}{\text{cl}(p)} + \frac{\|\lambda\|}{\text{cl}(\lambda)}
\]

We have

\[
\frac{d}{dx} \mu(\gamma(p, \lambda)) = -\frac{x_{\beta_T} - x}{x^2 \tan \theta_\kappa} = \frac{x \tan \theta_\kappa - x_{\beta_T}}{x^2 \tan \theta_\kappa}
\]

This expression is negative for \( x \) near 0, positive for large \( x \), and it has at most one root within feasible values of \( x \), namely at \( x = x_{\beta_T}/\tan \theta_\kappa \). Therefore, \( \mu(\gamma(p, \lambda)) \) is minimized when either \( \text{cl}(\lambda) = \text{cl}(p) \) or \( x = x^* = \min\{x_{\beta_T}/\tan \theta_\kappa, x_{\alpha_T}, x_{\beta_T}\} \). We pick \( w^*_\kappa \) so that \( \text{cl}(w^*_\kappa) = x^* \tan \theta_\kappa \).

**Case 2b** Suppose \( \kappa_T \) is a parabolic arc, and suppose \( q \) lies on \( \kappa_T \). Without loss of generality, the parabola supporting \( \kappa_T \) is equidistant between \( o \) and a point located at \((0, 2y_\kappa)\). Therefore, the parabola is described by the equation \( y = x^2/(4y_\kappa) + y_\kappa \). Define \( x \) and choose \( \lambda \) as in Case 2a. Again, we say \( x \) is feasible if \( \text{cl}(\lambda) \geq \text{cl}(p) \) and \( x_{\alpha_T} \leq x \leq x_{\beta_T} \).

Recall Corollary 2.1. Restricting ourselves to feasible values of \( x \), we have

\[
\mu(\gamma(p, \lambda)) = \ln \frac{\text{cl}(\lambda)}{\text{cl}(p)} + \frac{\|\lambda\|}{\text{cl}(\lambda)}
\]

We have

\[
\frac{d}{dx} \mu(\gamma(p, \lambda)) = -\frac{x_{\beta_T} - x}{x^2 \tan \theta_\kappa} = \frac{x \tan \theta_\kappa - x_{\beta_T}}{x^2 \tan \theta_\kappa}
\]
We have
\[ \frac{d}{dx} \mu(y(p, \lambda)) = \frac{2x^3 + 4y_x x^2 + 8y_{xx} (y_{xx} - x_{\beta+}) x - 16y_x^3}{(x^2 + 4y_x^2)^2}. \]

This expression is negative for \( x \) near 0 and positive for large \( x \). The derivative of the numerator is \( 6x^2 + 8y_x x + 8y_{xx} (y_{xx} - x_{\beta+}) \), which has at most one positive root. Therefore, the numerator has at most one positive local maximum or minimum. We see \( \frac{d}{dx} \mu(y(p, \lambda)) \) goes from negative to positive around exactly one positive root (which may not be feasible), and \( \mu(y(p, \lambda)) \) has one minimum at a positive value of \( x \). Let \( x' \) be this root of \( \frac{d}{dx} \mu(y(p, \lambda)) \). Value \( \mu(y(p, \lambda)) \) is minimized when either \( \text{cl}(\lambda) = \text{cl}(p) \) or \( x = x' = \min \{ \text{max} \{ x', x_{\lambda+} \}, x_{\beta+} \} \). We pick \( w_{\star} \) so that \( \text{cl}(w_{\star}) = (x')^2/(4y_x) + y_x \).

Case 2c) Suppose \( y \) lies on \( \alpha_T \). Let \( y \) be any value such that \( 0 \leq y \leq \text{cl}(u_T) \). Suppose we choose \( \lambda \) such that \( \lambda \) has clearance \( y \). As mentioned, the optimal choice for \( y \) guarantees \( y \geq \text{cl}(p) \). We say \( y \) is feasible if \( y \geq \text{cl}(p) \) and \( 0 < y \leq \text{cl}(u_T) \).

Recall Corollary 2.1. Restricting ourselves to feasible values of \( y \), we have
\[ \mu(y(p, \lambda)) = \ln \frac{\text{cl}(\lambda)}{\text{cl}(p)} + \frac{\|\lambda\|}{\text{cl}(\lambda)} = \ln \frac{y}{\text{cl}(p)} + \frac{x_{\beta+} - x_{\alpha \gamma}}{y}. \]

We have
\[ \frac{d}{dy} \mu(y(p, \lambda)) = \frac{1}{y} - \frac{(x_{\beta+} - x_{\alpha \gamma})}{y^2}. \]

This expression is negative for \( y \) near 0, positive for large \( y \), and it has at most one root within feasible values of \( y \), namely at \( y = x_{\beta+} - x_{\alpha \gamma} \). If \( \text{cl}(p) \) and \( x_{\beta+} - x_{\alpha \gamma} \) are at most \( \text{cl}(u_T) \), then \( \mu(p, \lambda) \) is minimized when either \( \text{cl}(\lambda) = \text{cl}(p) \) or \( y = y^* = x_{\beta+} - x_{\alpha \gamma} \). If either \( \text{cl}(p) \) or \( x_{\beta+} - x_{\alpha \gamma} \) are greater than \( \text{cl}(u_T) \), then it costs less for \( \lambda \) to intersect \( \kappa_T \) than for it to intersect \( \alpha_T \). Therefore, if \( y^* \leq \text{cl}(u_T) \), we pick \( w_{\star} \) so that \( \text{cl}(w_{\star}) = y^* \). Otherwise, we set \( w_{\star} = w_{2}^\gamma \).

\[ \square \]

4 Approximation Algorithms

In this section, we propose a near-quadratic-time \( (1 + \varepsilon) \)-approximation algorithm for computing the minimal-cost path. We first give a high-level overview of the algorithm and then describe each step in detail. Throughout this section, let \( \gamma^* \) denote a minimal-cost \((s,t)\)-path.

**High-level description.** Our algorithm begins by computing the refined Voronoi diagram \( \hat{V} \) of \( \mathcal{O} \). The algorithm then works in three stages. The first stage computes an \( O(n) \)-approximation of \( d^* = \mu(s,t) \), i.e., it returns a value \( \hat{d} \) such that \( d^* \leq \hat{d} \leq \varepsilon n d^* \) for some constant \( c > 0 \). By augmenting \( \hat{V} \) with a linear number of additional edges, each a constant-clearance path between two points on the boundary of a cell of \( \hat{V} \), the algorithm constructs a graph \( G_1 \) with \( O(n) \) vertices and computes a minimal-cost path from \( s \) to \( t \) in \( G_1 \).

Equipped with the value \( \hat{d} \), the second stage computes an \( O(1) \)-approximation of \( d^* \). For a given \( d \geq 0 \), this algorithm constructs a graph \( G_2 \) by sampling \( O(n) \) points on the boundary of each cell \( T \) of \( \hat{V} \) and connecting these sample points by adding \( O(n) \) edges (besides the boundary of \( T \)), each of which is again a constant-clearance path. The resulting graph \( G_2 \) is planar, so a minimal-cost path in \( G_2 \) from \( s \) to \( t \) can be computed in \( O(n^2) \) time \([9]\). We show that if \( d \geq \hat{d} \), then the cost of the optimal path from \( s \) to \( t \) in \( G_2 \) is \( O(d) \). Therefore, if \( d \in [d^*/2, 2d^*] \), the cost of the optimal path is \( O(d^*) \). Using the value of \( \hat{d} \), we run the above procedure for \( O(\varepsilon n) \) different values of \( d \), namely \( d \in \{ d^*/2 | 0 \leq i \leq \varepsilon n \} \), and return the least costly path among them. Let \( \hat{d} \) be the cost of the path returned.

Finally, using the value \( \hat{d} \), the third stage samples \( O(n/\varepsilon^2) \) points on the boundary of each cell \( T \) of \( \hat{V} \) and connects each point to \( O((1/\varepsilon) \log n/\varepsilon) \) other points on the boundary of \( T \) by an edge. Unlike the last two stages, each edge is no longer a constant-clearance path but it is a minimal-cost path between its endpoints lying inside \( T \). The resulting graph \( G_3 \) has \( O(n^2/\varepsilon^2) \) vertices and \( O((n^2/\varepsilon^2) \log(n/\varepsilon)) \) edges. The overall algorithm returns the minimal-cost path in \( G_3 \).

The analysis of all three stages relies on the guarantees of Lemma 3.2 concerning the existence of anchor points and well-behaved paths between points on the boundary of each cell of \( \hat{V} \).

4.1 \( O(n) \)-approximation algorithm

Here, we describe a near-linear time algorithm to obtain an \( O(n) \)-approximation of \( d^* \). We augment \( \hat{V} \) with \( O(n) \) additional edges as described below to create the graph \( G_1 \).

We do the following for each cell \( T \) of \( \hat{V} \). We compute anchor points \( w_{\alpha \gamma}^* \) and \( w_{\alpha \gamma}^\alpha \) as described in Lemma 3.2. We subdivide \( \beta_T \) at \( w_{\alpha \gamma}^* \) and \( w_{\alpha \gamma}^\alpha \), and add constant-clearance line segments or circular arcs \( \lambda_{\alpha \gamma} \) and \( \lambda_{\alpha \gamma} \) from \( w_{\alpha \gamma}^* \) and \( w_{\alpha \gamma}^\alpha \), respectively, to the other points of equal clearance on the edges.
of $T$. Finally, let $w_s$ be the point on $\beta_T$ of clearance min\{c($v_T$), c($s$)$\}$. We subdivide $\beta_T$ at $w_s$ and add the constant clearance path $\lambda_\ast$ from $w_s$ to the other point of equal clearance on $T$’s edges. Note that some of these paths may be trivial if they begin at $v_T$. See Figure 9. All edges in $G_1$ are assigned the cost of their path using (1.1) and the equations of Wein et al. [17] for Voronoi edges. We compute and return the minimal-cost path in $G_1$ from $s$ to $t$.

**Lemma 4.1.** Graph $G_1$ contains an $s,t$-path of cost at most $O(n) \cdot d^*$.  

**Proof.** Suppose $\gamma^\ast$ has points outside $G_1$ in the interior of Voronoi cell $T$. We use the notation given above for adding edges to $G_1$ within $T$. Let $p$ and $q$ be the first and last intersection of $\gamma^\ast$ and $T$. We construct a well-behaved $(p,q)$-path $\gamma$ through $G_1$ such that $\mu(\gamma) = O(d^*)$. If neither $p$ nor $q$ lie on $\beta_T$, then $\gamma$ follows the edges of $T$ in $\tilde{\mathcal{V}}$. See Lemma 3.1.

Suppose otherwise, and let $p$ lie on $\beta_T$. Suppose $q$ lies on $\alpha_T$. Let $w$ be a point of higher clearance between $w_w^\ast$ and $w_s$, and let $\lambda$ be the path of higher clearance between $\lambda_w^\ast$ and $\lambda_s$. Path $\gamma$ follows $\beta_T$ from $p$ to $w_s$, follows $\beta_T$ from $w_s$ to $w$, follows $\lambda$, and then follows $\kappa_T$ to $q$. Note that $\gamma$ may use some points of $\beta_T$ twice; we describe it as we do to simplify the analysis.

By Corollary 2.1, $\mu(p,w_s) \leq d^*$. Therefore, $\mu(w_s,q) \leq \mu(w_s,p) + \mu(p,q) \leq 2d^*$. From Lemma 3.2, we have $\mu(\gamma|w_s,q) \leq 7\mu(w_s,q) = O(d^*)$. Finally, $\mu(\gamma) = \mu(p,w_s) + \mu(\gamma|w_s,q) = O(d^*)$. A similar construction is used if $q$ lies on $\alpha_T$.

We replace $\gamma^\ast[p,q]$ with $\gamma$, reducing the number of cells containing points in $\gamma^\ast$ disjoint from $G_1$. After repeating this procedure in $O(n)$ different Voronoi cells, we create a path through $G_1$ with cost $O(n) \cdot d^*$.  

Diagram $\tilde{\mathcal{V}}$ contains $O(n)$ vertices, edges, and cells. Graph $G_1$ contains a constant number of additional vertices and edges per cell, so it has $O(n)$ vertices and edges total. Computing the minimal-cost $s,t$-path in $G_1$ takes $O(n \log n)$ time.  

**Theorem 4.2.** Let $O$ be a set of polygonal obstacles in the plane, and let $s,t$ be two points outside $O$. There exists an $O(n \log n)$-time $O(n)$-approximation algorithm for computing the minimal-cost path between $s$ and $t$.

---

4.2 Constant-factor approximation

Recall that, given an estimate $d$ of the cost $d^\ast$ of the optimal path, we construct a planar graph $G_2$ by sampling points along the edges of the refined Voronoi diagram $\tilde{\mathcal{V}}$. The sampling procedure here can be thought of as a warm-up for the sampling procedure given in Section 4.3.

Let $T$ be a Voronoi cell of $\tilde{\mathcal{V}}$ and assume without loss of generality that $c(s) \leq c(t)$. Let $w_{\min}$ and $w_{\max}$ be the points on $\beta_T$ with clearance $c(l(t))/\exp(d)$ and $\min\{c(v_T),c(s) \cdot \exp(d)\}$ respectively. We place sample points on $\beta_T$ between $w_{\min}$ and $w_{\max}$ inclusive to act as vertices in $G_2$. The samples are chosen so that the cost between consecutive samples is exactly $\frac{d}{2}$ (except possibly at one endpoint). Given a sample point $p$ on an edge of $\tilde{\mathcal{V}}$, it is straightforward to compute the coordinates of the sample point $p'$ on the same edge such that $c(p,p') = c$ for any $c > 0$. Simply use the formula for the cost along a Voronoi edge given in [17, Corollary 8]. We emphasize that the points are separated evenly by cost; the samples will not be uniformly placed in terms of the Euclidean distance along the edge. Figure 6 shows the samples used for our constant-factor approximation algorithm as well as our third algorithm described below.

From each sample point, we add a constant-clearance edge to $G_2$ within $T$ to the other point on the edges of $T$ with the same clearance, subdividing the edges as necessary. We also add constant clearance edges within $T$ from anchor points $w_w^\ast$ and $w_s^\ast$ as defined in Lemma 3.2. The refined Voronoi diagram $\tilde{\mathcal{V}}$ is planar. Every edge added to create $G_2$ stays within a single cell of $\tilde{\mathcal{V}}$ and has constant clearance. Therefore, no pair of new edges cross and $G_2$ is planar as well. We compute the minimal-cost path from $s$ to $t$ in $G_2$ using the linear (in graph size) time algorithm of Henzinger et al. [9].

**Lemma 4.3.** Suppose $d \geq d^\ast$ and $c_l(s) \leq c_l(t)$. Value $c(l(t))/\exp(d)$ is a lower bound on the minimal clearance attained by $\gamma^\ast$ and $c_l(s) \cdot \exp(d)$ is an upper bound on the maximal clearance attained by $\gamma^\ast$.

**Proof.** Let $p_{\min}$ be the point where $\gamma^\ast$ attains the minimal clearance. Clearly, $\mu(s,t) \geq \mu(s,p_{\min}) + \mu(p_{\min},t)$. Using this observation together with Corollary 2.1, we obtain our lower bound. The upper bound follows by similar arguments.

For each Voronoi cell $T$, let $\hat{\beta}_T$ be the portion of $\beta_T$ that receives sample points. We have the following two properties: (i) cost $\mu(\hat{\beta}_T) = O(d)$ and (ii) if $d \geq d^\ast$, then no point on $\beta_T \setminus \hat{\beta}_T$ can lie on $\gamma^\ast$.
Therefore, the total number of vertices added along each edge is $O(n)$.

**Property (i) follows by (2.4). Property (ii) follows from Lemma 4.3.**

**Lemma 4.4.** Suppose $d \geq d^*$. Graph $G_2$ contains an $s, t$-path of cost $O(d)$. 

**Proof.** Let $\gamma$ be a maximal portion of $\gamma^*$ lying in a single Voronoi cell $T$ of $\mathcal{V}$, and let $p$ and $q$ be the endpoints of $\gamma$. If neither $p$ nor $q$ lies on $\beta_T$, then Lemma 3.1 guarantees that the edges of $T$ contain a well-behaved path $\gamma'$ of cost at most $7\mu(\gamma)$. Suppose otherwise, and let $p$ lie on $\beta_T$ without loss of generality. By property (ii), there exists a sample point $p'$ on $\beta_T$ such that $\mu(p, p') \leq \frac{d}{n}$. We have $\mu(p', q) \leq \mu(p', p) + \mu(p, q) \leq \mu(p, q) + \frac{d}{n}$. By Lemma 3.2 and the choice of edges in $G_2$, there exists a well-behaved path in $G_2$ through $T$ of cost at most $7\mu(p', q)$; if this path enters the interior of $T$, then it does so at one of $p'$, $w^*_n$, or $w^*_\beta$. In particular, there exists a path $\gamma'$ in $G_2$ from $p$ to $q$ of cost at most $7\mu(p, q) + O(\frac{d}{n})$.

Each edge of $\mathcal{V}$ is a minimal-cost path. Therefore, each edge is incident to at most two maximal subpaths of $\gamma^*$ internally disjoint from $\mathcal{V}$. We conclude there are $O(n)$ such subpaths. Each can be replaced by one going through $G_2$ as described above. The total cost of the new path from $s$ to $t$ is $7d^* + O(n) \cdot O(\frac{d}{n}) = O(d)$. \hfill \Box

The refined Voronoi diagram $\mathcal{V}$ contains $O(n)$ vertices and edges. Property (i) ensures that the number of vertices added along each edge $e$ is $O(n)$. Therefore, the total number of vertices added along edges of $\mathcal{V}$ is $O(n^2)$. We use a linear-time algorithm [9] to compute a minimal-cost path in planar graph $G_2$, so constructing $G_2$ and finding a minimal-cost path $G_2$ takes $O(n^2)$ time.

For our constant-factor approximation algorithm, we perform an exponential search over the values of path costs. Let $\hat{d} \leq cn d^*$ be the cost of the path returned by the $O(n)$-approximation algorithm (Section 4.1). For each $i$ from 0 to $\lceil \log cn \rceil$, we take $d = \hat{d}/2^i$ as the estimate of $d^*$, and run the above procedure to construct a graph $G_2$ and compute a minimal-cost path in the graph. We return the least costly of the paths computed over all iterations.

Fix integer $i$ so $d^* \leq \hat{d}/2^i \leq 2d^*$. Let $d_i$ be the cost of the minimal-cost path in $G_2$ during iteration $i$. Let $d$ be the minimal output of the $O(1)$-approximation algorithm over the set of $O(\log n)$ iterations. By Lemma 4.4, we have

$$d \leq d_i \leq O(\hat{d}/2^i) = O(d^*).$$

**Theorem 4.5.** Let $\mathcal{O}$ be a set of polygonal obstacles in the plane, and let $s, t$ be two points outside $\mathcal{O}$. There exists an $O(n^2 \log n)$ time $O(1)$-approximation algorithm for computing the minimal-cost path between $s$ and $t$.

### 4.3 Computing the final approximation

Finally, let $\hat{d}$ be the estimate returned by our constant factor approximation algorithm so that $d^* \leq \hat{d} \leq cd^*$ for some constant $c$. We construct a graph $G_3$ by sampling points along the edges of the refined Voronoi diagram $\mathcal{V}$.

**Sample vertices in $G_3$.** Let $T$ be a Voronoi cell$^6$ of $\mathcal{V}$ and assume without loss of generality that $\text{cl}(s) \leq \text{cl}(t)$. In each case below, points within a single region are sampled so they lie at cost $\frac{cd}{n}$ apart. Along $\alpha_T$, $\beta_T$,

---

$^6$Note that as we consider each cell independently, we actually consider each edge $e$ twice as it is adjacent to two cells. However, as follows from the description, the same set of samples are placed on the edge regardless of which cell one considers. Considering each edge twice does not change the complexity of the algorithm or its analysis and simplifies the description.
we place samples between points $w_{\alpha T}^{\min}$ and $w_{\alpha T}^{\max}$ with clearance $c(t)/\exp(\delta)$ and $\min\{\text{cl}(w_T), \text{cl}(s) \cdot \exp(\delta)\}$, respectively (including at $w_{\alpha T}^{\min}$ and $w_{\alpha T}^{\max}$). Along $\beta_T$, we place samples between points $w_{\beta T}^{\min}$ and $w_{\beta T}^{\max}$ with clearance $c(t)/\exp(\delta)$ and $\min\{\text{cl}(w_T), \text{cl}(s) \cdot \exp(\delta)\}$, respectively. Along $\kappa_T$, we place samples between $u_T$ and the point on $\kappa_T$ of cost $2\delta$ from $u_T$. Additionally, let $v'$ be the point of clearance $\min\{\text{cl}(s) \cdot \exp(\delta), \text{cl}(w_T)\}$ on $\kappa_T$. We place samples on $\kappa_T$ between $v'$ and the point $u'$ on $\kappa_T$ of cost $4\delta$ from $v'$ such that $\text{cl}(u') \leq \text{cl}(v')$. See Figure 6.

The edges of $G_3$. Let $T$ be a cell of $\tilde{V}$ incident to obstacle feature $o$. We say two points $p$ and $q$ in $T$ are locally reachable from one another if the minimal-cost path from $p$ to $q$ relative only to $o$ lies within $T$. Equivalently, the minimal-cost path relative to $o$ is equal to the minimal-cost path relative to $\emptyset$.

Let $p$ be a sample point on $\beta_T$. We compute anchor point $w_o^*$ as described in Lemma 3.2. We then compute a collection of sample points $S(p)$ on $\kappa_T$ as candidate neighbors of $p$ in $G_3$. Let $u_T$ be the portion of $\kappa_T$ that receives sample points, and let $\eta$ denote one of the (possibly overlapping) connected regions of equally spaced sample points on $u_T$ as described above. Region $\eta$ is either the set of points on $\kappa_T$ within cost $2\delta$ from $w_T$, or the set of points on $\kappa_T$ lying between $u'$ and $v'$. See Figures 6 and 7. We add the following points to $S(p)$. For each $w \in \{p, w_o^*\}$ (lying on $\beta_T$), let $\downarrow (w)$ be the sample point on $\kappa_T$ of highest clearance less than $\text{cl}(w)$ in $\eta$ (assuming such a point exists). Let $\uparrow (w)$ be the sample point on $\kappa_T$ of lowest clearance greater than $\text{cl}(w)$ in $\eta$. Let $i_{\max}$ be the greatest $i$ such that $(1 + \epsilon)^i$ is at most the number of sample points on $\eta$.

To begin, we add to $S(p)$ the endpoints of $\eta$. For each $q_0 \in \{\downarrow (p), \downarrow (w_o^*)\}$, we add the following points to $S(p)$. First, we add $q_0$ to $S(p)$. We then iteratively walk along each sample point of $\eta$ in decreasing order of clearance starting with $q_0$. For each non-negative integer $i \leq i_{\max}$, we add the point $q_i$ encountered at step $[(1 + \epsilon)^i]$ of the walk. Similarly, for each $q_0 \in \{\uparrow (p), \uparrow (w_o^*)\}$, we add to $S(p)$ the point $q_0$ and perform the walk along points of greater clearance. See Figure 7.

We add edges from $p$ to each locally reachable point $q$ of $S(p)$. The cost of the edge $(p, q)$ is the cost of the optimal path from $p$ to $q$ relative to the feature $o$, as given in (2.2) or (2.3).

We add a similar set of points from each $\eta$ of $\kappa_T$ or $\alpha_T$ to $S(p)$ as well as edges to those locally reachable members of $S(p)$. Finally, a similar procedure exists for sample points $p$ on $\alpha_T$. For such $p$, set $S(p)$ contains points on $\kappa_T$ and edges are added to locally reachable members of that set. We compute the minimal-cost path from $s$ to $t$ in $G_3$ using Dijkstra’s algorithm with Fibonacci heaps [8].

Let $\epsilon$ be an edge of $\tilde{V}$ and recall that $\hat{e}$ denotes the portion of $e$ that receives sample points. We will show that the following two properties (analogous to the two properties stated in Section 4.2) hold: (i) cost $\mu(\eta) = O(\hat{d})$ for each connected component $\eta$ of $\hat{e}$ and (ii) if $d \geq d^*$, then no point on $e \setminus \hat{e}$ can lie on $\gamma^*$. Property (i) holds by our choice of sampling intervals and Corollary 2.1. We now show property (ii) holds.

Let $T$ be a cell bounded by $e$, and let $\gamma$ be any $s,t$-path that passes through $T$ such that $\mu(\gamma) \leq \hat{d}$ (assuming one exists). Let $p$ and $q$ be the points where $\gamma$ enters and exits $T$, respectively. We will show that $q$ lies within the regions sampled.

If $q$ lies on an internal arc then $\text{cl}(t)/\exp(\delta) \leq \text{cl}(q) \leq \text{cl}(s) \cdot \exp(\delta)$. The claim holds. Suppose $q$ lies on the external arc $\kappa_T$. If $p$ lies on $\alpha_T$, then by the analysis of Case 1 in Lemma 3.1, $\mu(u_T, q) \leq 2\mu(\gamma) \leq 2\hat{d}$. Point $q$ lies in a sampled region. If not, then $p$ lies on $\beta_T$. Following the lines of the analysis of Case
2 in Lemma 3.1, \( \mu(v', q) \leq 4\mu(\gamma) \leq 4\tilde{d} \). Again, \( q \) lies in a sampled region.

We have the following lemmas which will allow us to prove that using \( G_3 \), one can compute a \((1 + \varepsilon)\)-approximation to the minimal-cost path problem.

**Lemma 4.6.** Let \( T \) be a cell of \( \tilde{V} \) incident to obstacle feature \( o \). Let \( p \) be a point on \( T \)'s edges, and let \( e_0 \) be an edge of \( T \) not containing \( p \). Let \( Q_p \) be the set of points on \( e_0 \) locally reachable from \( p \). If \( Q_p \) is non-empty, then it is connected and has one boundary at an endpoint of \( e_0 \).

**Proof.** We consider two main cases.

**Case 1** Suppose \( o \) is a polygon vertex. Without loss of generality, \( o \) lies at the origin, edge \( \alpha_T \) intersects the line \( y = 0 \) at the origin with angle \( \theta_{\alpha_T} \). Line \( \beta_T \) intersects the line \( y = 0 \) at the origin with angle \( \theta_{\beta_T} \), and \( \theta_{\beta_T} > \theta_{\alpha_T} \geq 0 \). Let \( q \in Q_p \). If \( q \) does not exist, then there is nothing to prove. Otherwise, let \( \gamma \) be the minimal-cost path from \( p \) to \( q \). Equation (2.2) describes the cost of \( \gamma \). We consider a mapping \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) taking points to a transformed plane. Given in polar coordinates point \( (r, \theta) \), the mapping \( f \) is defined as \( f(r, \theta) = (\theta, \ln r) \). Given a path \( \gamma' \), we abuse notation and let \( f(\gamma') = f \circ \gamma' \), the composition of \( f \) and \( \gamma' \). Path \( \gamma \) becomes a straight-line segment connecting \( p \) and \( q \) in the transformed plane. Both \( \alpha_T \) and \( \beta_T \) become vertical rays in the transformed plane going to \( -\infty \). Further, it is straightforward to show that \( \kappa_T \) becomes a convex curve in the transformed plane when restricted to values of \( \theta \) such that \( \theta_{\alpha_T} \leq \theta \leq \theta_{\beta_T} \). Two points \( p' \) and \( q' \) in \( T \) are locally reachable from another if and only if the line segment between \( f(p') \) and \( f(q') \) does not cross the boundary of \( T \) in the transformed plane (see Figure 8a).

Suppose \( p \) lies on \( \alpha_T \) (Figure 8b). Point \( u_T \) is clearly locally reachable from \( p \). Let \( L_p \) be the set of lines in the transformed plane that intersect the point \( f(p) \). The transformed line segment connecting \( f(p) \) to \( f(u_T) \) is supported by the (vertical) line \( \ell_{\infty} \in L_p \) with infinite slope. Consider ordering the lines in \( \ell \in L_p \) by decreasing slope starting with \( \ell_{\infty} \). The first intersection of each \( \ell \in L_p \) with \( f(\kappa_T) \) moves farther to the right as the slope decreases. Moreover, each of these first intersections are locally reachable from \( p \). This is true until some line \( \ell^* \) goes tangent to \( f(\kappa_T) \) at \((\theta^*, \ln r^*)\), and no line intersects \( f(\kappa_T) \) again after that point. Line \( \ell^* \) is the first line to intersect \( \beta_T \) without first crossing \( f(\kappa_T) \). If \( e_0 = \kappa_T \), then \( Q_p \) consists of all points \( q \) such that \( f(q) \) lies on \( f(\kappa_T) \) between \( f(u_T) \) and \((\theta^*, \ln r^*)\). If \( e_0 = \beta_T \), then \( Q_p \) consists of all points \( q \) such that \( f(q) \) lies on \( \beta_T \) below the intersection of \( f(\beta_T) \) and \( \ell^* \). A similar argument holds if \( p \) lies on \( \beta_T \).

Finally, suppose \( p \) lies on \( \kappa_T \) and \( e_0 = \alpha_T \) (Figure 8c). Let \( \ell^* \) be the line tangent to \( f(\kappa_T) \) at \( f(p) \). Line \( \ell^* \) intersects \( f(\alpha_T) \) at \( f(w) \). Lines to higher points on \( f(\alpha_T) \) that intersect \( f(p) \) cross \( f(\kappa_T) \), so \( w \) is the highest clearance point on \( \alpha_T \) locally reachable from \( p \). All lower points on \( f(\alpha_T) \) are locally reachable, though. Therefore, \( Q_p \) consists of all points \( q \) on \( \alpha_T \) of clearance at most \( c_l(w) \). A similar argument holds if \( p \) lies on \( \kappa_T \) and \( e_0 = \beta_T \).

**Case 2** Suppose \( o \) is a polygon edge. Without loss of generality, \( o \) lies on the line \( y = 0 \), the edge \( \alpha_T \) lies on the line \( x = x_{\alpha_T} \), the edge \( \beta_T \) lies on the line \( x = x_{\beta_T} \), and \( x_{\beta_T} > x_{\alpha_T} \geq 0 \). There is no notion of the transformed plane for polygon edge \( o \), but we are still able to use similar arguments to those given in Case 1. Two points \( p' \) and \( q' \) in \( T \) are locally reachable from one another if the circular arc containing \( p' \) and \( q' \) centered on \( o \) does not cross \( \kappa_T \).

Suppose \( p \) lies on \( \alpha_T \) (see Figure 9a). Point \( u_T \) is clearly locally reachable from \( p \). Let \( L_p \) be the set of circles that intersect the point \( p \) and have their center on \( o \). The line segment connecting \( p \) to \( u_T \) is supported by the degenerate circle \( C_{\infty} \in L_p \) with center at \((\infty, 0)\). The upper semi-circle of \( C_{\infty} \) contains the point \((x, \infty)\) for every \( x > x_{\alpha_T} \). Consider ordering the circles of \( L_p \) by decreasing x-coordinate of their centers, starting with \( C_{\infty} \). The upper semi-circle of each \( C \in L_p \) is a concave curve, and edge \( \kappa_T \) is a convex curve; the upper semi-circles all intersect \( \kappa_T \) at most two times. Fix an \( x > x_{\alpha_T} \). The point \((x, y)\) on the upper semi-circle of each \( C \in L_p \) moves downward as the circle centers move left until one of the circles intersects \((x, 0)\). Indeed, the bisector of line segment \( p(x, y) \) must continue intersecting the center of each \( C \in L_p \) as the centers move left. Therefore, there is some last \( C^* \in L_p \) that intersects \( \kappa_T \); circle \( C^* \) and \( \kappa_T \) are tangent at point \( w^* = (x^*, y^*) \).

The first circles \( C \in L_p \) in our ordering lie above \( \kappa_T \) at each x-coordinate, but eventually they lie below \( \kappa_T \) at each x-coordinate. Also, the second crossing of any \( C \in L_p \) and \( \kappa_T \) can never occur to the left of \( w^* \). We conclude that each point \((x, y)\) on \( \kappa_T \) between \( u_T \) and \( w^* \) must be locally reachable from \( p \). If \( e_0 = \kappa_T \), then these points are precisely \( Q_p \).

Now, assume some point on \( \beta_T \) is locally reachable from \( p \). Circle \( C^* \) is the first to reach \( \beta_T \) without crossing \( \kappa_T \). As the center of each \( C \in L_p \) moves left from \( C^* \)'s center, the intersection of \( C \) and \( \beta_T \) moves downward. If \( e_0 = \beta_T \) and \( Q_p \) is non-empty, we have \( Q_p \) consisting of all points on \( \beta_T \) between \( o \) and the intersection of \( C^* \) and \( \beta_T \). A similar set of
arguments hold if \( p \) lies on \( \beta_T \).

Finally, suppose \( p \) lies on \( \kappa_T \) and \( e_0 = \alpha_T \) (see Figure 9b). Let \( C^* \) be the circle centered on \( o \) which lies tangent to \( \kappa_T \) at \( p \). Circle \( C^* \) contains a point \( q^* \) on \( \alpha_T \) locally reachable from \( p \). If we initially take \( C = C^* \) and move the center of \( C \) to the left, then the arc between \( p \) and \( \alpha_T \) along \( C \) moves above \( \kappa_T \) and must cross again closer to \( \alpha_T \). However, if we move the center of \( C \) to the right, then the arc between \( p \) and \( \alpha_T \) stays below \( \kappa_T \). In addition, the intersection of \( C \) and \( \alpha_T \) moves downward. Therefore \( Q_p \) consists of all points \( q \) such that \( q \) lies on \( \alpha \) below \( q^* \). A similar argument holds if \( p \) lies on \( \kappa_T \) and \( e_0 = \beta_T \).

**Lemma 4.7.** Graph \( G_3 \) contains an \( s,t \)-path of cost at most \((1 + O(\varepsilon))d^*\).

**Proof.** Let \( \gamma \) be a maximal subpath of \( \gamma^* \) lying interior to a cell \( T \) of \( \mathcal{V} \) and let \( p \) and \( q \) be the endpoints of \( \gamma \). Let \( e_p \) be the edge of \( T \) containing \( p \) and \( e_q \) be the edge of \( T \) containing \( q \). By Corollary 2.1, \( e_p \neq e_q \). We assume \( p \) lies on \( \beta_T \) and \( q \) lies on \( \kappa_T \). The other cases are the same. Points \( p \) and \( q \) are locally reachable from each other. By Lemma 4.6 and property (ii) given above, there exists a sample point \( p' \) locally reachable from \( q \) on \( \beta_T \) such that \( \mu(p,q') \leq \frac{\varepsilon d}{n} \). We have \( \mu(p',q') \leq \mu(p,q) + \frac{\varepsilon d}{n} \). Suppose there exists a point \( q' \in S(p') \) on \( \kappa_T \) locally reachable from \( p' \) such that \( \mu(q,q') \leq \frac{\varepsilon d}{2n} \). In this case, there exists a path from \( p \) to \( q \) through \( G_3 \) which takes edge \( p'q' \) and has total cost at most \( \mu(p,q) + \frac{4\varepsilon d}{n} \).

Suppose there is no locally reachable \( q' \) as described above. Lemma 3.2 describes how to find a well-behaved path \( \gamma' \) between \( p' \) and \( q \) such that \( \mu(\gamma') \leq \frac{7}{10} \mu(p,q) \).

There exists \( \lambda = \lambda(\gamma') \) with one endpoint on \( \kappa_T \). Let \( w' \) be the endpoint of \( \lambda \) on \( \kappa_T \). Path \( \gamma' \) follows \( \kappa_T \) from \( w' \) to \( q \). Recall our algorithm adds sample points along several regions of length \( O(\hat{d}) \) such that each pair of points lies at cost \( \frac{\varepsilon d}{n} \) apart. Point \( q \) lies in one of these regions \( \eta \). By assumption, \( q \) is at least \( \frac{\varepsilon d}{n} \) cost away from any sample point of \( \eta \cap S(p') \).
Therefore, \( w' \) and \( q \) cannot both lie between a pair of consecutive sample points on \( \eta \). Let \( q_0 \) be the first sample point of \( \eta \) encountered by \( \gamma^* \) on \( s \tau \). Path \( \lambda \) has an endpoint on \( \kappa \tau \) of clearance \( \text{cl}(p') \) or \( \text{cl}(w'_*) \). Therefore, \( q_0 \in \{ \downarrow (p'), \downarrow (w'_*), \uparrow (p'), \uparrow (w'_*) \} \).

For each of these possible \( q_0 \), our algorithm adds samples \( q_i \), \( 0 \leq i \leq \ell \), spaced geometrically away from \( q_0 \) in the direction of \( q \). These samples include one endpoint of \( \eta \). Let \( q_k \) be the last of these sample points closer to \( q_0 \) than \( q \), and let \( q_{k+1} \) be the next of these sample points. By Lemma 4.6, at least one of \( q_k \) and \( q_{k+1} \) is locally reachable from \( p \). Let \( q' \) be this locally reachable point.

Let \( \delta = \mu(q_0, q) \frac{2}{d} \). Value \( \delta \) is an upper bound on the number of samples in \( \eta \) between \( q_0 \) and \( q \). We have \( \left( 1 + \delta \right)^k \leq \delta \leq \left( 1 + \frac{\delta}{d} \right)^k \). In particular \( \delta \leq (1 + \frac{\delta}{d})^{k+1} \), which implies \( \delta \leq \left( 1 + \frac{\delta}{d} \right)^k \leq \varepsilon d + 1 \). Similarly, \( (1 + \frac{\delta}{d})^k - \delta \geq \varepsilon \delta \). Also, \( \mu(q_0, q) \leq 7 \mu(p', q) \).

\[
\mu(q, q') \leq \left( \varepsilon d + 1 \right) \frac{\varepsilon d}{n} \\
\leq \left( \mu(q_0, q) \frac{\varepsilon n}{\varepsilon d} + 1 \right) \frac{\varepsilon d}{n} \\
= \varepsilon \mu(q_0, q) + \frac{\varepsilon d}{n} \\
\leq 7 \varepsilon \mu(p', q) + \frac{\varepsilon d}{n}.
\]

We have \( \mu(p', q') \leq \mu(p', q) + \mu(q, q') \leq (1 + 7 \varepsilon) \cdot \mu(p', q) + \frac{\varepsilon d}{n} \). There exists a path from \( p \) to \( q \) through \( G_3 \) which takes edge \( p'q' \) and has total cost \( (1 + 7 \varepsilon) \cdot \mu(p, q) + O(\frac{\varepsilon d}{n}) \).

Each edge of \( \hat{V} \) is a minimal-cost path. Therefore, each edge is incident to at most two maximal subpaths of \( \gamma^* \) internally disjoint from \( \hat{V} \). We conclude there are \( O(n) \) such subpaths. Each can be replaced by one going through \( G_3 \) as described above. The total cost of the new path from \( s \) to \( t \) is

\[
(1 + 7 \varepsilon) \cdot d^* + O(n) \cdot O(\frac{\varepsilon d}{n}) = \\
(1 + O(\varepsilon)) \cdot d^* + O(\varepsilon) \cdot \frac{\varepsilon d}{n} = \\
(1 + O(\varepsilon))d^*.
\]

Recall that \( n \) denotes the number of obstacle features. The refined Voronoi diagram \( \hat{V} \) contains a linear number of vertices and edges. Vertices are sampled at intervals of cost \( \frac{\varepsilon d}{n} \). Therefore, property (i) ensures that the number of vertices added along each edge \( e \) is \( O(\frac{\varepsilon d}{n}) \). In turn, the total number of vertices in \( G_3 \) is \( O(\frac{\varepsilon d}{n}) \). Each sample vertex on an internal edge is incident to \( O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}) \) edges of \( G_3 \), bringing the total number of edges in \( G_3 \) to \( O(\frac{\epsilon^2 d}{n}) \). Recall, we compute the minimal-cost path from \( s \) to \( t \) using Dijkstra’s algorithm with Fibonacci heaps [8].

Thus, we obtain the following theorem.

**Theorem 4.8.** Let \( \mathcal{O} \) be a set of polygonal obstacles in the plane with \( n \) vertices total, and let \( s, t \) be two points outside \( \mathcal{O} \). Given a parameter \( \varepsilon \in (0, 1) \), there exists an \( O((\frac{\epsilon^2 d}{n}) \log \frac{1}{\varepsilon}) \)-time approximation algorithm for the minimal-cost path problem between \( s \) and \( t \) such that the algorithm returns an \( s, t \)-path of cost \( (1 + \varepsilon) d^* \).

5 **Discussion**

In this paper we present the first polynomial-time approximation algorithm for the problem of computing minimal-cost paths between two given points (when using the cost defined in (1.1)). Our immediate goal is to improve the running time of our algorithm to be near-linear. A possible approach would be to refine the notion of anchor points so it suffices to put only \( O(\log n) \) additional points on each edge of the refined Voronoi diagram.

Finally, there are natural interesting open problems that we believe should be addressed. The first is to determine if the problem at hand is NP-Hard. When considering the complexity of such a problem, one needs to consider both the algebraic complexity and the combinatorial complexity. In this case we believe that the algebraic complexity may be high because of the cost function we consider. However, we believe that combinatorial complexity, defined analogously to the number of “edge sequences”, may be small. We are currently investigating if this is indeed the case. The second natural interesting open problem calls for extending our algorithm to compute near-optimal paths amid polyhedral obstacles in \( \mathbb{R}^3 \).

**References**


Copyright © by SIAM.

Unauthorized reproduction of this article is prohibited.