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Sensor Based Planning, Part I: The Generalized Voronoi Graph

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Abstract. *This paper introduces a 1-dimensional network of curves termed the* Generalized Voronoi Graph *(GVG) and its extension, the* Hierarchical Generalized Voronoi Graph *(HGVG), which can be used as a basis for a roadmap or retract-like structure. The GVG and HGVG provide a basis for sensor based path planning in an unknown static environment. In this paper, the GVG and HGVG are defined and some of their properties are exploited to show their utility for motion planning. A companion paper describes how to use the GVG and HGVG for the purposes of sensor based planning.*

1 Introduction

Sensor Based Planning incorporates sensor information, reflecting the current state of the environment, into a robot's planning process, as opposed to *Classical Planning*, which assumes full knowledge of the world's geometry prior to planning. This paper and its companion [7] introduce a sensor based motion planning scheme that is useful for two closely related motion planning problems: (1) to determine the path which connects two points in a robot's free space, or determine such a path does not exist; and (2) to build a concise "map" which encodes the important topological information about the robot's free space. The method, which is based on "retract-like" structures termed the *Generalized Voronoi Graph* (GVG), and its extension, the *Hierarchical Generalized Voronoi Graph* (HGVG), requires only local sensor information to construct the motion plan. That is, no a priori knowledge of the robot's environment is assumed. This paper considers only point or spherical robots. However, we believe that these techniques can be extended to more general cases.

The primary goal of this first paper is to introduce the GVG and HGVG and their properties. While our intention is to use the GVG and HGVG as a basis for sensor based planning, they can also be used for classical motion planning when full knowledge of the world's geometry is available. The companion paper describes an incremental technique for constructing the GVG from local sensor data. Further, the companion paper provides experimental results which validate the method.

2 Relation to Prior Work

Sensor based planning has received increased attention, as it is a requirement for realistic deployment of autonomous robots in unstructured environments. For a review of many sensor-based planning techniques, see [16]. Unfortunately, current sensor based planning methods are limited because: (1) many are based on heuristic algorithms, and it is therefore impossible to prove if they will work in all possible environments; or (2) proof of convergence is limited to the case of 2-dimensional environments (for example, Lumelsky's "bug" algorithm [10]). The goal of this work is to develop provably correct motion planning schemes for workspace dimensions greater than two, and which can be robustly implemented with realistic sensors.

Our approach is to adapt the structure of a rigorous motion planning scheme to a sensor based implementation. There are three classes of complete motion planning schemes: cellular decomposition methods, potential field approaches, and retract or roadmap methods [9]. Roadmaps or retract-like structures capture the global topological properties of the robot's free space and have the following important properties: *accessibility*, *departability* and *connectivity*. These properties imply that the planner can construct a path between any two points in a connected component of the robot's free space by first finding a path onto the roadmap (accessibility), traversing the roadmap to the vicinity of the goal (connectivity), and then constructing a path from the roadmap to the goal (departability).

The Generalized Voronoi Diagram (GVD) (i.e., a Voronoi Diagram for the case in which the sites are sets, and not points) was first used for motion planning in [14]. Active research in applying Voronoi Diagrams to motion planning began with [11], which considered motion planning for a disk in the plane. However, the method in [11] requires full knowledge of the world's geometry prior to the planning event; and its retract methodology may not extend to non-planar problems. In [12], an incremental approach to create a Voronoi Diagram-like structure, which is limited to the case of a plane, was introduced. In the companion paper, we investigate an incremental algorithm to construct the GVG and HGVG using only local sensor data. Further, our method can be used for non-planar problems.

To our knowledge, the first complete sensor based adaption of a roadmap motion planning scheme for workspace dimension greater than two, was introduced by Rimon [13]; it was Rimon's method which has motivated our work. Rimon's approach is a sensor based extension of Canny and Lin's Opportunistic Path Planner (OPP) [4]. From a practical point of view, there are two detractions to Rimon's method. First, to construct the roadmap, the robot must contain "interesting critical point" and "minimum passage" sensors, whose implementation is not well described. Second, a robust and detailed procedure for constructing the roadmap fragments from sensor data is not presented. We choose instead to base our sensor based planning scheme on a different structure, which we term the Generalized Voronoi Graph (GVG) and the Hierarchical Generalized Voronoi Graph (HGVG). We have found these structures to be easier to construct using realistic sensors. Second, we are able to give a rigorous procedure for robustly constructing the graph components from sensor data.

The GVG introduced in this paper appears to be new, though a GVG-like structure for SE(3) is described in [3]. In prior work (e.g., [2]) the Voronoi Graph has only been defined for point sites, whereas this work extends the Voronoi Graph concept to the case of set sites. In dimensions greater than 2, the GVG is not connected. The other contribution of this paper is a scheme for connecting the GVG in these cases. Another important contribution of this work is the definition of the GVD and GVG in terms of distance functions. By using this alternative definition, methods from differential topology and nonsmooth analysis can be applied to the analysis of the GVD and the GVG. Furthermore, it can be shown that sensors readily provide distance information, thus making our definition of the GVD, GVG and HGVG amenable to sensor based implementation.

3 Distance Functions

A function which encodes the distance between the robot and nearby obstacles is key to our definitions. This section defines a distance function, and its gradient.A more complete discussion of these functions and their properties can be found in [5]. We assume a point robot operating in a subset, $\mathcal{W}$, of an m-dimensional Euclidean space. $\mathcal{W}$ is populated by obstacles $C_1, \cdots, C_n$ which are convex sets. Non-convex obstacles are modeled as the union of convex shapes. It is assumed that the boundary of $\mathcal{W}$ is a collection of convex sets, which are members of the obstacle set $\{C_i\}$.

Definition 3.1 *Single Object Distance Function.* The distance between a point, x and a convex set C_i is

$$d_i(x) = \min_{c_0 \in C_i} \|x - c_0\|, \tag{1}$$

where $\|\cdot\|$ is the 2-norm in $\mathbb{R}^m$. In [8] it is shown that the gradient of $d_i(x)$ is

$$\nabla d_i(x) = \frac{x - c_0}{\|x - c_0\|}. \tag{2}$$

Thus, $\nabla d_i(x)$ is a unit vector in the direction from c_0 to x, where c_0 is the nearest point to x in C_i. For convex sets, the closest point is always unique. **An important characteristic of $d_i(x)$ and $\nabla d_i(x)$ is that they can be computed from sensor data.**

4 The Generalized Voronoi Graph

This section defines the Generalized Voronoi Diagram and the Generalized Voronoi Graph via the above distance functions. The basic building block of the GVD and GVG is the set of points equidistant to two sets C_i and C_j, which we term the *Two-Equidistant Surface*, $\mathcal{S}_{ij}$.

$$\mathcal{S}_{ij} = \{x \in \mathbb{R}^m : d_i(x) - d_j(x) = 0\} \tag{3}$$

Of particular interest is the subset of $\mathcal{S}_{ij}$ termed the *Two-Equidistant Surjective Surface*, $\mathcal{SS}_{ij}$:

$$\mathcal{SS}_{ij} = \{x \in \mathcal{S}_{ij} : \nabla d_i(x) \neq \nabla d_j(x)\}. \tag{4}$$

These are the set of points equidistant to two objects such that $\nabla d_i \neq \nabla d_j$, i.e., the function $(d_i - d_j)(x)$ is surjective. This definition is necessary to deal with non-convex sets that are defined as the finite union of convex sets. If $\mathcal{W}$ is solely populated with disjoint convex sets, then $\mathcal{SS}_{ij} = \mathcal{S}_{ij}$, $\forall\, i, j$.

The *Two-Equidistant Face*, $\mathcal{F}_{ij}$, is the set of points equidistant to obstacles C_i and C_j, such that each point $x \in \mathcal{SS}_{ij}$ is closer to C_i and C_j than any other obstacle.

$$\mathcal{F}_{ij} = \{x \in \mathcal{SS}_{ij} : d_i(x) \leq d_k(x) \quad \forall k \neq i, j\} \tag{5}$$

A Two-Equidistant Face is also termed a *Generalized Voronoi Face* in keeping with the conventions of the Voronoi Diagram literature. The relationship between Eqs.3, 4, and 5 is shown in Fig. 1. The *Two-Voronoi Set*, $\boldsymbol{\mathcal{F}}^2$, is the union of all Two-Equidistant Faces.

$$\boldsymbol{\mathcal{F}}^2 = \bigcup_{i=1}^{n-1} \bigcup_{j=i+1}^{n} \mathcal{F}_{ij} \tag{6}$$

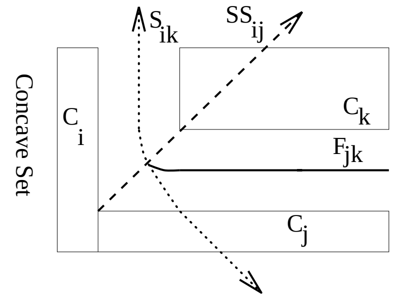

Fig. 1. Solid lines = $\mathcal{F}_{jk}$, Dashed lines $\mathcal{SS}_{ij}$, Dotted line, $\mathcal{S}_{ik}$

Since $\boldsymbol{\mathcal{F}}^2$ is the set of points equidistant to two or more obstacles, it can be shown that $\boldsymbol{\mathcal{F}}^2$ is the Generalized Voronoi Diagram.

To define the GVG, we continue to define lower dimensional subsets of $\mathcal{W}$. The *Three-Equidistant Face*, $\mathcal{F}_{ijk}$, is the set of points equidistant to C_i, C_j and C_k such that each point is closer to C_i, C_j, and C_k than any other object. Similarly, the *Three-Voronoi Set* is the union of all the Three-Equidistant Faces.

$$\begin{aligned} \mathcal{F}_{ijk} &= \mathcal{F}_{ij} \bigcap \mathcal{F}_{ik} \bigcap \mathcal{F}_{jk} = \mathcal{F}_{ij} \bigcap \mathcal{F}_{ik} \\ \boldsymbol{\mathcal{F}}^3 &= \bigcup_{i=1}^{n-2} \bigcup_{j=i+1}^{n-1} \bigcup_{k=j+1}^{n} \mathcal{F}_{ijk} \end{aligned} \tag{7}$$

Note, $\mathcal{SS}_{ijk}$ and $\mathcal{S}_{ijk}$ can be defined in a similar manner such that $\mathcal{F}_{ijk} \subset \mathcal{SS}_{ijk} \subset \mathcal{S}_{ijk}$. Continuing in this vein, after taking $k-2$ intersections, one can define a *k-Equidistant Face*, $\mathcal{F}_{i_1 i_2 \ldots i_k}$ which is the set of points equidistant to objects $C_{i_1}, C_{i_2}, \ldots C_{i_k}$, such that each point is closer to objects $C_{i_1}, C_{i_2}, \ldots C_{i_k}$ than any other object. The *k-Voronoi Set* is simply the union of k-Equidistant Faces.

$$\begin{aligned} \mathcal{F}_{i_1 i_2 \ldots i_k} &= \mathcal{F}_{i_1 i_2} \bigcap \mathcal{F}_{i_1 i_3} \cdots \bigcap \mathcal{F}_{i_1 i_k} \\ &= \mathcal{F}_{i_1 i_2 \ldots i_{k-1}} \bigcap \mathcal{F}_{i_1 i_k} \\ \boldsymbol{\mathcal{F}}^k &= \bigcup_{i_1=1}^{n-k+1} \bigcup_{i_2=i_1+1}^{n-k+2} \cdots \bigcup_{i_k=i_{k-1}+1}^{n} \mathcal{F}_{i_1 i_2 \ldots i_k} \end{aligned} \tag{8}$$

Again, $\mathcal{SS}_{i_1 i_2 \ldots i_k}$ and $\mathcal{S}_{i_1 i_2 \ldots i_k}$ can be defined in a similar manner where $\mathcal{F}_{i_1 i_2 \ldots i_k} \subset \mathcal{SS}_{i_1 i_2 \ldots i_k} \subset \mathcal{S}_{i_1 i_2 \ldots i_k}$. Furthermore, it can be shown that $\mathcal{F}_{i_1 i_2 \ldots i_{k+1}} \subset \partial \mathcal{F}_{i_1 i_2 \ldots i_k}$.

In m dimensions, the *Generalized Voronoi Edge* and *Generalized Voronoi Vertex* are respectively an m-Equidistant Face, $\mathcal{F}_{i_1 \ldots i_m}$ and $(m+1)$-Equidistant Face, $\mathcal{F}_{i_1 \ldots i_{m+1}}$. It will be shown that the Generalized Voronoi Edge is 1-dimensional, while the Generalized Voronoi Vertex is a point where Generalized Voronoi Edges meet. Generalized Voronoi Vertices are sometimes called *meet points* because that is where Generalized Voronoi Edges "meet." Using these definitions, we can define the Generalized Voronoi Graph.

Definition 4.1 *The Generalized Voronoi Graph* (GVG) is defined to be the collection of all of the Generalized Voronoi Edges, and Generalized Voronoi Vertices of a bounded space.

$$GVG = (\boldsymbol{\mathcal{F}}^m, \boldsymbol{\mathcal{F}}^{m+1}) \tag{9}$$

The GVG's edges are the set of points equidistant to m objects, such that each point is closer to m objects than any other object. **An important characteristic of the GVG is that it is defined in terms of the distance functions, which can be readily computed from sensor data.** For subsequent analysis, it is useful to define the following.

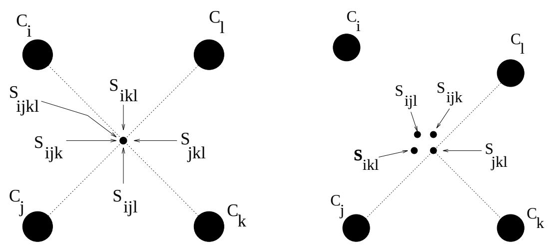

Fig. 2. (L) Nongeneric arrangement (R) Small perturbation

DEFINITION 4.2 *The Generalized Voronoi Region,* $\mathcal{F}_i$, is the set of points closer to one particular object than any other object.

$$\mathcal{F}_i = \{x \in \mathbb{R}^m : d_i(x) \leq d_j(x) \quad \forall j \neq i\} \tag{10}$$

It can be shown that the $\mathcal{F}_i$ is generalized star shaped, i.e., $\forall x \in \mathcal{F}_i$, there is a closest point $c_i \in C_i$ to x such that the line $\overline{xc_i}$ is fully contained in $\mathcal{F}_i$. That is, there is a straight line in free space between any point in $\mathcal{F}_i$ and some point on C_i.

5 Basic Properties of the GVG

To determine the generic dimension of the edges, we will use the Pre-image Theorem below to show that the intersection of a k-Equidistant Face and a 2-Equidistant Face is $(k-1)$-dimensional. In order to properly invoke the Pre-image Theorem, we must make the following transversality assumption.

ASSUMPTION 5.1 (The Equidistant Surface Transversality Assumption): *We assume that equidistant surfaces intersect transversally. That is,* $\mathcal{S}_{i_1 \ldots i_k j_1} \pitchfork \mathcal{S}_{i_1 \ldots i_k j_2}$ *with respect to* $\mathcal{S}_{i_1 \ldots i_k}$ *if and only if* $j_1 \neq j_2$.

In the case that $m = 2$ and the obstacles are points, this assumption is equivalent to the "no four points are co-circular" assumption which is often made in the Voronoi Diagram literature. Assumption 5.1 is the generalization of this statement, and shows more rigorously why such assumptions arise. This transversality assumption can also be interpreted as an assumption on the stability of the equidistant surface intersection geometry. In the left diagram of Fig. 2, $\mathcal{S}_{ijk} = \mathcal{S}_{jkl} = \mathcal{S}_{ikl} = \mathcal{S}_{ijl}$ because there exists a circle which is tangent to the 4 obstacles (a non-generic case). After a slight perturbation of the obstacles, the Equidistant Surfaces no longer coincide (Fig 2). Since $\mathcal{S}_{ijk}$ and $\mathcal{S}_{ijl}$ are points in this example, they intersect transversally only if they do not intersect at all. The following is a corollary to Assumption 5.1, and its proof is omitted.

COROLLARY 5.2 *(The Equidistant Surface Uniqueness Result):* $\mathcal{S}_{i_1 \ldots i_k j_1} \neq \mathcal{S}_{i_1 \ldots i_k j_2}$ iff $j_1 \neq j_2$.

To show that the edges are 1-dimensional, we invoke the Pre-image Theorem [1] $m-1$ times on the difference of two distance functions. We first introduce the following two lemmas.

LEMMA 5.3 $\mathcal{F}^2$, $\mathcal{F}_{ij}$, and $\mathcal{SS}_{ij}$ have co-dimension 1 in $\mathbb{R}^m$.
Proof: First, note that the function $(d_i - d_j)(x)$ is smooth [8] by the obstacle convexity assumption. Recall that the 2-Equidistant Surjective Surface, $\mathcal{SS}_{ij}$, is a subset of $\mathcal{S}_{ij}$ such that $\nabla d_i(x) \neq \nabla d_j(x)\ \forall x \in \mathcal{SS}_{ij}$. This implies that $\nabla(d_i - d_j)(x)$ is surjective; thus 0 is a regular value of $(d_i - d_j)$ on $\mathcal{SS}_{ij}$. By the Pre-image Theorem, since 0 is a regular value of the smooth function $(d_i - d_j)(x)$, $\mathcal{SS}_{ij}$ is a manifold having co-dimension 1 in $\mathbb{R}^m$. Since $\mathcal{F}_{ij}$ is a subset of $\mathcal{SS}_{ij}$ of the same dimension, it too is a co-dimension one set. $\mathcal{F}^2$ is a co-dimension 1 set (not necessarily a manifold) because it is the finite union of co-dimension 1 sets. ▼

LEMMA 5.4 $\mathcal{F}^3$, $\mathcal{F}_{ijk}$ and $\mathcal{SS}_{ijk}$ each have co-dimension 2 in $\mathbb{R}^m$.
Proof: $\mathcal{SS}_{ijk}$ can also be defined as $\mathcal{SS}_{ijk} = \{x \in \mathcal{SS}_{ij} : d_i(x) - d_k(x) = 0\}$. By Corollary 5.2, $\mathcal{SS}_{ij} \neq \mathcal{SS}_{ik} \iff i \neq k$. Therefore, 0 is a regular value of $(d_i - d_k)(x)$ on $\mathcal{SS}_{ij}$. By the Pre-image Theorem, $\mathcal{SS}_{ijk}$ is co-dimension 1 in $\mathcal{SS}_{ij}$, and thus co-dimension 2 in $\mathbb{R}^m$. $\mathcal{F}_{ijk}$ is a subset of $\mathcal{SS}_{ijk}$ and thus is co-dimension 2 in $\mathbb{R}^m$. Since $\mathcal{F}^3$ is the finite union of co-dimension 2 manifolds, it has co-dimension 2 in $\mathbb{R}^m$. ▼

By induction, one can show that the set of points equidistant to k obstacles has co-dimension 1 in the set of points equidistant to $k-1$ objects, and therefore this set has co-dimension $k-1$ in $\mathbb{R}^m$. Hence, $\mathcal{SS}^m$ is 1-dimensional in $\mathbb{R}^m$ and since $\mathcal{F}^m \subset \mathcal{SS}^m$, the GVG edges are 1-dimensional. By a similar argument, the vertices, $\mathcal{F}^{m+1}$, are zero-dimensional. This proves the following proposition.

PROPOSITION 5.5 The Generalized Voronoi Edges of the GVG, $\mathcal{F}^m$, are 1-dimensional in $\mathbb{R}^m$, and the Generalized Voronoi Vertices of the GVG, $\mathcal{F}^{m+1}$, are points.

Another key feature of the GVG is that it provides a concise representation of the robot's free space.

6 Accessibility/Departability

Accessibility is the property that a path can be constructed from any point in the free space to the Generalized Voronoi Graph. In this section, we give a very simple argument that a path exists from any point in the free space to a GVG edge.

PROPOSITION 6.1 The GVG has the property of accessibility.
Proof: Let x_k be a point on a k-Equidistant Face, $\mathcal{F}_{i_1,\ldots,i_k}$, and x_{k+1} be a point on the (k+1)-Equidistant Face, $\mathcal{F}_{i_1,\ldots,i_{k+1}}$, which is on the boundary of $\mathcal{F}_{i_1,\ldots,i_k}$. In an m-dimensional world, for $2 \leq k \leq m$, there always exists a a collision-free paths constrained to a k-Equidistant Face between x_k and x_{k+1} because for $k \geq 2$, k-Equidistant Faces are a subset of the free space. Therefore, by re-invoking the above statement, there exists a collision-free path from any k-Equidistant Face to the GVG.

It can be shown [6], [11] that there always exists a collision-free path from any point in the free space to a 2-Equidistant Face. Therefore, from any arbitrary point in the free space, there exists a collision-free path to the GVG. ■

Departability can be shown to be accessibility, but in reverse. However, in the companion paper, we introduce an algorithm for the departing process.

7 Connectivity of the GVG

The Generalized Voronoi Regions and Equidistant Faces may be viewed as a cellular decomposition of $\mathcal{W}$ into k-dimensional sets, where $k = 0, \ldots, m$. If each k-dimensional cell is homeomorphic to a k-dimensional disk, then the 1-dimensional cells of such a decomposition form a deformation retract or retract-like structure of $\mathcal{W}$ [15]. Equivalently, if the boundary of each k-dimensional closed cell is connected, then

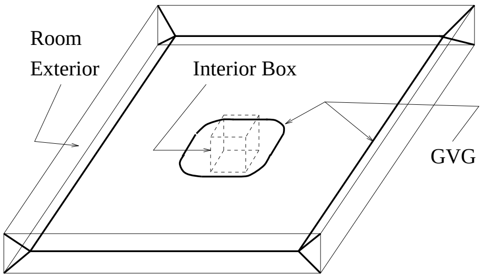

Fig. 3. An example of a disconnected GVG

the resulting one dimensional cells form a retract-like structure of $\mathcal{W}$ [6]. One of the appealing properties of a retract-like structure is its connectivity.

For $m = 2$ (i.e., planar environments), the GVD and the GVG are the same. It is well known in this case that the planar GVD is connected. However, $m > 2$, the GVG is not necessarily connected. The GVG-like structure for SE(3) is described in [3] also suffers from the problem of connectivity not being guaranteed. Fig. 3 shows an example where $\mathcal{W}$ is a box, with one box-like obstacle in the interior. For some dimensions of the box obstacle and enclosure, the GVG will be disconnected while for other sizes of the box and enclosure, the GVG will be connected.

In the next section we introduce the notion of Higher Order Generalized Voronoi Graphs. These will be used to link disconnected components of the GVG by subdividing higher dimensional Equidistant Faces (k-dimensional cells) via a tessellation into closed regions whose boundaries are connected (or readily link up). For example, when $\mathcal{W} \subset \mathbb{R}^3$, the problem reduces to linking up disconnected boundaries (i.e., the Generalized Voronoi Edges) of a 2-Equidistant Face via a tessellation into closed two dimensional regions whose boundaries are connected. This method reduces a higher dimensional problem into a 2-dimensional problem, which is more tractable.

8 The Second Order GVG

This section defines the *Second Order Generalized Voronoi Graph*, GVG^2 which is defined on a 2-Equidistant Face, $\mathcal{F}_{ij}$. The Second Order Generalized Voronoi Graph is the set of points on a 2-Equidistant Face that are "second closest" to nearby obstacles. The basic building block of the Second Order GVG is the *Second Order 2-Equidistant Face*

$$\mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}} = \{x \in \mathcal{F}_{ij} : (d_l - d_k)(x) = 0 \quad and \quad \forall h,\ d_h(x) \geq d_k(x) = d_l(x) \geq d_i(x) = d_j(x)\} \tag{11}$$

It is the set of points where C_k and C_l are the *second* closest equidistant objects and C_i and C_j are the closest equidistant objects.

One additional structure needs to be defined. Let the set of points on the boundary of a k-Equidistant Face which are equidistant to k obstacles be termed the *k-Boundary Edge*, defined by $\mathcal{C}_{i_1 \ldots i_k}$. This occurs either when $d_{i_1}(x) = \cdots = d_{i_k}(x) = 0$ or when $\nabla d_p(x) = \nabla d_q(x)$ for $p, q \in \{i_1, \ldots, i_k\}$. Most k-Equidistant Faces do *not* have k-Boundary Edges; when all of the obstacles are disjoint and convex, there are no k-Boundary Edges.

The *Second Order 2-Voronoi Set*, which is also the *Second Order GVD* is

$$\boldsymbol{\mathcal{F}}^2\big|_{\mathcal{F}_{ij}} = \bigcup_k \bigcup_l \mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}} \left(\bigcup \mathcal{C}_{ij}\right) \tag{12}$$

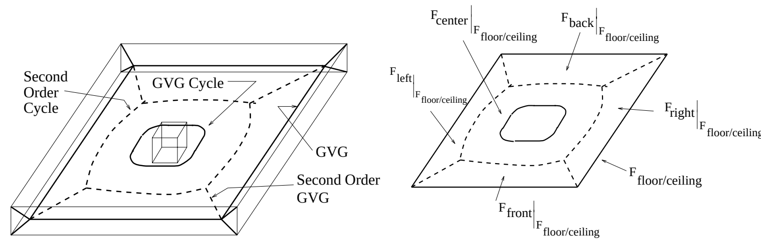

Fig. 4. Second Order GVG

Analogous to the GVG, we continue our construction with lower dimensional subsets of $\mathcal{F}_{ij}$. The *Second Order Three-Equidistant Face,* $\mathcal{F}_{klp}\big|_{\mathcal{F}_{ij}}$, is the set of points where C_k, C_l and C_p are *second* closest equidistant objects and C_i and C_j are the closest equidistant objects. Furthermore, the *Second Order 3-Voronoi Set* is the union of all the 3-Equidistant Faces.

$$\begin{aligned} \mathcal{F}_{klp}\big|_{\mathcal{F}_{ij}} &= \mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}} \bigcap \mathcal{F}_{lp}\big|_{\mathcal{F}_{ij}} \bigcap \mathcal{F}_{kp}\big|_{\mathcal{F}_{ij}} \\ \boldsymbol{\mathcal{F}}^3\big|_{\mathcal{F}_{ij}} &= \bigcup_k \bigcup_l \bigcup_p \mathcal{F}_{klp}\big|_{\mathcal{F}_{ij}} \end{aligned} \tag{13}$$

The Second Order k-Equidistant Face is simply the intersection of the appropriate $k - 3$ Second Order 2-Equidistant Faces. In m-dimensions, the *Second Order Generalized Voronoi Edge* is some Second Order $m - 1$-Equidistant Face. Note, it is defined by $m - 3$ intersections whereas the (First Order) Generalized Voronoi Edge is defined by $m - 2$ intersections. It can be easily shown by the Pre-image Theorem that the edges of the GVG^2 are 1-dimensional. Finally, the *Second Order Generalized Voronoi Vertex* is a Second Order m-Equidistant Face, and it is zero-dimensional.

Definition 8.1 (Second Order GVG) (Constrained to a 2-Equidistant Face)

$$GVG^2\big|_{\mathcal{F}_{ij}} = (\boldsymbol{\mathcal{F}}^{m-1}\big|_{\mathcal{F}_{ij}}, \boldsymbol{\mathcal{F}}^m\big|_{\mathcal{F}_{ij}})$$

is the set of points equidistant to the second closest $m - 1$ objects such that C_i and C_j are the closest equidistant objects. Note that a Second Order GVG may not exist on every $\mathcal{F}_{ij}$.

The definition of Higher Order GVG's follows accordingly, and thus we get the following definition:

Definition 8.2 (Hierarchical Gen. Voronoi Graph) The *Hierarchical Generalized Voronoi Graph* (HGVG) is the union of the Generalized Voronoi Graph and all higher order Generalized Voronoi Graphs.

$$\text{HGVG} = \text{GVG} \bigcup_{i=2}^{i=m-1} \text{GVG}^i$$

For subsequent analysis, it will be useful to define the *Second Order Generalized Voronoi Region,* $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$, which is the set of points constrained to a 2-Equidistant face, $\mathcal{F}_{ij}$, whose second closest object is C_k.

$$\mathcal{F}_k\big|_{\mathcal{F}_{ij}} = \{x \in \mathcal{F}_{ij} \quad s.t. \quad \forall h \neq i, j, k \quad d_i(x) = d_j(x) \leq d_k(x) \leq d_h(x)\} \tag{14}$$

The GVG$^2\big|_{\mathcal{F}_{ij}}$ divides up $\mathcal{F}_{ij}$ into Second Order Generalized Voronoi Regions. See Fig. 4 for an example of a GVG^2 and how $\mathcal{F}_{floor/ceiling}$ is divided up into Second Order Generalized Voronoi Regions.

When the obstacles satisfy a certain condition (defined in Section 10), the $\bigcup_i$ GVG^i will link the disconnected components of the GVG. This condition is a constraint which guarantees that no GVG cycles (GVG edges diffeomorphic to the unit circle) may exist. In Figure 4, the GVG^2 does not link up the GVG because of the existence of GVG cycles. The following section carefully analyzes cycles, and the section after that states the conditions under which cycles do not exist. It will be shown that when cycles do not exist, all of the k-Equidistant Faces are divided up into regions whose boundaries are all connected. In this case, the HGVG connects all disconnected GVG fragments.

9 Cycles

To simplify the discussion, we focus only on the case of $m = 3$, where HGVG = GVG $\bigcup$ GVG^2. However, analogous methods exist for $m > 3$ [6]. In this section, we carefully analyze cycles, so that in the next section, we can state the condition under which they do not exist. First we define cycles and show that they lead to GVG fragments which are disconnected from both other GVG edges and GVG^2 edges.

Definition 9.1 (GVG Cycle) is a Generalized Voronoi Edge which is diffeomorphic to S^1, the unit circle.

Henceforth, the term "cycle" refers to a GVG cycle.

Proposition 9.2 A GVG edge is a cycle if and only if it is disconnected from the GVG and the GVG^2.

Proof: This proof is a simple consequence of the following Lemma whose proof appears in the Appendix.

Lemma 9.3 A Second Order Generalized Voronoi Edge can only intersect the GVG at a meet point.

GVG cycles do not contain meet points, and thus GVG edges and GVG^2 edges can not intersect them. That is, they are disconnected. In a bounded space, the only disconnected GVG edges are cycles. ∎

Proposition 9.4 A GVG edge is a disconnected component of a boundary of a Second Order Generalized Voronoi Region if and only if it is a cycle.

Proof: The proof of this now employs the following three lemmas whose proofs appear in the Appendix.

Lemma 9.5 If a GVG edge, $\mathcal{F}_{ijk}$, exists then that implies the existence of a Second Order Generalized Voronoi Region, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ on the Two-Equidistant Face, $\mathcal{F}_{ij}$, and $\mathcal{F}_{ijk}$ is a subset of the boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$.

Lemma 9.6 (Uniqueness) There can be at most *one* GVG edge, $\mathcal{F}_{ijk}$, on the boundary of a Second Order Generalized Voronoi Region, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$.

Lemma 9.7 On a 2-Equidistant face which has more than one GVG edge on its boundary, there will always be a GVG^2 on that 2-Equidistant Face.

By Lemma 9.5, the cycle $\mathcal{F}_{ijk}$ must be a subset of the boundary of a Second Order Generalized Voronoi Region, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. Second Order Generalized Voronoi Edges (Lemma 9.7) and perhaps Boundary Voronoi Edges (by definition) are the other structures which *may* exist on the boundary of a Second Order Generalized Voronoi Region. Since by Proposition 9.2, neither of these can intersect the GVG cycle, $\mathcal{F}_{ijk}$ must lie on a disconnected portion of the boundary of a Second Order Generalized Voronoi Region.

And now for the converse, if $\mathcal{F}_{ijk}$ is a disconnected boundary component of a Second Order Generalized Voronoi Region, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$, it is a GVG cycle. Again, if $\mathcal{F}_{ijk}$ is disconnected, it does not intersect a GVG edge, nor GVG^2 edge. By Proposition 9.2, $\mathcal{F}_{ijk}$ is a cycle. ∎

Second Order Cycles. Just as there is a cycle in the Generalized Voronoi Graph, there are also cycles in the Second Order Generalized Voronoi Graph. In order to define the GVG^2 cycle, we need to recall the definition of the Second Order Generalized Voronoi Region, which is based on the definition of the Generalized Voronoi Region.

Definition 9.8 (GVG^2 Cycle) is a cycle comprised of GVG^2 edges, and perhaps a fragment of a Boundary Edge, which solely forms a connected component of the boundary of a Second Order Generalized Voronoi Region.

A GVG^2 cycle is written as $\bigcup_l \mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}}$ or $\bigcup_l \mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}} \bigcup c_{ij}$ where $c_{ij} \subset C_{ij}$ is a fragment of the Boundary Edge, C_{ij}.

The following proposition shows there is a duality between the existence of GVG and GVG^2 cycles. In order for one of them to exist, then the other must exist. If no GVG cycles exist, then there can not be any GVG^2 cycles, and visa versa.

Proposition 9.9 Let $\mathcal{F}_{ij}$, $\mathcal{F}_{ik}$ and $\mathcal{F}_{jk}$ be three Two-Equidistant Faces whose intersection forms $\mathcal{F}_{ijk}$. If the GVG edge, $\mathcal{F}_{ijk}$ is a cycle, then on at least one of the Two-Equidistant Faces, $\mathcal{F}_{ij}$, $\mathcal{F}_{ik}$, and $\mathcal{F}_{jk}$ there exists a second order cycle. The converse is also true — if there exists a Second Order Cycle, and there is a Generalized Voronoi Edge associated with it, then the Generalized Voronoi Edge is a cycle.

Proof: The existence of $\mathcal{F}_{ijk}$ implies the existence of $\mathcal{F}_{ij}$, $\mathcal{F}_{ik}$ and $\mathcal{F}_{jk}$. By Proposition 9.2, if $\mathcal{F}_{ijk}$ is a cycle, then it is a disconnected boundary component on each: $\mathcal{F}_{ij}$, $\mathcal{F}_{ik}$ and $\mathcal{F}_{jk}$. Even though it is possible that $\mathcal{F}_{ijk}$ may be the only boundary component of a Two-Equidistant Face, by boundedness $\mathcal{F}_{ijk}$ can not be the sole boundary component on all three Two-Equidistant Faces. Therefore, at least one of the Two Equidistant Faces, say $\mathcal{F}_{ij}$, has another boundary component — implying that, $\partial\mathcal{F}_{ij} = \bigcup_p \mathcal{F}_{ijl_p} \bigcup \mathcal{F}_{ijk} \left(\bigcup c_{ij}\right)$.

By Lemma 9.5, the existence of $\mathcal{F}_{ijk}$ implies $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ exists on $\mathcal{F}_{ij}$ such that $\mathcal{F}_{ijk}$ is on the boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. By Lemma 9.6, $\mathcal{F}_{ijk}$ and any other Generalized Voronoi Edge, can not exist on any other boundary component of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. By Boundedness and Lemma 9.7, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ must have another boundary component fully comprised of Second Order Generalized Voronoi Edges, and perhaps Boundary Edge Fragments. Such a boundary component is a Second Order Cycle, by definition.

The second order cycle consists of GVG^2 edges (and perhaps Boundary Edges) which form a boundary component of a Second Order Generalized Voronoi Region, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. By hypothesis, there exists a GVG edge, $\mathcal{F}_{ijk}$ and thus by Lemma 9.5, it is on the boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. By Lemma 9.6, $\mathcal{F}_{ijk}$ is the only GVG edge inside of the Second Order Generalized Voronoi Region. Therefore by Lemma 9.3, no GVG or Second Order GVG Edges can emanate from $\mathcal{F}_{ijk}$. Since $\mathcal{F}_{ijk}$ is disconnected, by Proposition 9.4, $\mathcal{F}_{ijk}$ is a cycle because it is a disconnected boundary component of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. ∎

10 Extended Boundedness Assumption

With the above definitions and relationships in place, we are now able to state the assumption which will guarantee connectivity of the GVG $\bigcup$ GVG^2. This assumption will be used to eliminate environments in which cycles in the GVG arise.

Assumption 10.1 (Extended Boundedness) *For every combination of k equidistant obstacles ($2 \leq k \leq m$), there exists a point which is equidistant to a $k + 1^{st}$ obstacle.*

In $\mathbb{R}^3$ this means that all Two-Equidistant Faces contain at least one Generalized Voronoi Edge. Furthermore, all Generalized Voronoi Edges have at least one meet point. That is, $\forall i, j, k$, $\exists x \in \mathcal{F}_{ijk}, l$, such that $d_l(x) = d_k(x)$. By the Equidistant Surface Transversality Assumption (Ass. 5.1), this point is isolated.

By definition, this assumption is stronger in higher dimensional workspaces because it requires higher dimensional workspaces to be more "cluttered" than those of lower dimensions. Robots whose configuration spaces are higher tend to be highly articulated and are thus better suited for cluttered environments.

We will now show that the Extended Boundedness Assumption leads to a cycle-free environment. First, we will show that under the Extended Boundedness Assumption, all Second Order Generalized Voronoi Regions must have a Generalized Voronoi Edge on its boundary. This is necessary in showing that environments which satisfy the Extended Boundedness Assumption, do not have GVG nor GVG^2 cycles.

Lemma 10.2 Given the Extended Boundedness Assumption all Second Order Generalized Voronoi Regions must contain a Generalized Voronoi Edge.

Proof: Recall the definition of the Second Order Generalized Voronoi Region, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$.

$$\mathcal{F}_k\big|_{\mathcal{F}_{ij}} = \{x \in \mathcal{F}_{ij} : \forall h \notin \{i,j,k\}\, d_h(x) \geq d_k(x) = d_i(x)\}$$

The Extended Boundedness Assumption (Ass. 10.1), there exists some $h' \notin \{i, j\}$ and some x where such that $d_i(x) = d_j(x) = d_{h'}(x)$. If $h' = k$, then $\mathcal{F}_{ijk}$ exists, and by Lemma 9.5 and Lemma 9.6, it is the only Generalized Voronoi Edge in $\partial\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$.

If $h' \neq k$, then that implies, $\mathcal{F}_{ijh'}$ exists, that is, $d_i(x) = d_j(x) = d_{h'}(x)$. However, since the Second Order Generalized Voronoi Region $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ exists ($d_k(y) \leq d_{h'}(y)\, \forall y \in \mathcal{F}_k\big|_{\mathcal{F}_{ij}}$), by continuity of the single object distance function, $\mathcal{F}_{ijk}$ must also exist in $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ (Lemma 9.5). This however is a contradiction of Lemma 9.6, where only one GVG edge may exist in $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. Therefore $h' \neq k$, and $\mathcal{F}_{ijk}$ is always a subset of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. ▼

Lemma 10.3 Given Assumptions 5.1 and 10.1, there can not be any GVG, or GVG^2 cycles.

Proof: Let $\mathcal{F}_{ijk}$ be a Generalized Voronoi Edge. We will show it can not be a cycle. By the Extended Boundedness Assumption and Equidistant Surface Transversality Assumption, $\exists x \in \mathcal{F}_{ijk}$, such that $d_l(x) = d_k(x)$ which is isolated. That is $\mathcal{F}_{ijkl} = \mathcal{F}_{ijl} \bigcap \mathcal{F}_{ijk} \neq \emptyset$. The Equidistant Surface Transversality Assumption 5.1 guarantees that $\mathcal{F}_{ijl}$ and $\mathcal{F}_{ijk}$ intersect transversely. This rules out the possibility that $\mathcal{F}_{ijl}$ is tangent to $\mathcal{F}_{ijk}$, while $\mathcal{F}_{ijk}$ is diffeomorphic to S^1. Therefore, a GVG cycle can not exist.

By Proposition 9.9, if there exists: (1) a second order cycle, which is a component of the boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$, and (2) a Generalized Voronoi Edge which is a subset of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ (whose existence is guaranteed by the Extended Boundedness Assumption), then there exists a first order cycle.

The contrapositive of this statement is also true. If a GVG cycle does not exist, that implies a GVG^2 cycle can not exist *or* the Extended Boundedness Assumption is not in effect. The Extended Boundedness Assumption implies a GVG cycle can not exist which, in turn, implies a GVG^2 cycle can not exist *or* the Extended Boundedness Assumption is not in effect. However, since the Extended Boundedness Assumption is in effect, there can not be any GVG^2 cycles. ▼

11 Connectivity, Continued

And now we are ready to show that under the Extended Boundedness Assumption, the HGVG is connected. In [6], we show that if the union of the Second Order Generalized Voronoi Regions on a 2-Equidistant Face is the 2-Equidistant Face (trivial), and the boundaries of each of the Second Order Generalized Voronoi Regions are connected, then the Second Order Generalized Voronoi Graph connects disconnected GVG edge fragments on a Two-Equidistant Face. The following proposition shows that given the Extended Boundedness Assumption, the boundaries of each of the Second Order Generalized Voronoi Regions are connected.

Proposition 11.1 Given the Extended Boundedness Assumption, the Equidistant Surface Transversality Assumption, and the Boundedness Assumption, the boundary of a Second Order Generalized Voronoi Region is connected.

Proof: The boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ is comprised of Second Order Generalized Voronoi Edges, one Generalized Voronoi Edge (Lemma 10.2) and perhaps one or more Boundary Edge Fragments from the same Boundary Edge.

$$\partial\mathcal{F}_k\big|_{\mathcal{F}_{ij}} = \bigcup_{l \in L} \mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}} \bigcup \mathcal{F}_{ijk} \left(\bigcup c_{ij}\right)$$

where L is the set of indices, cataloging the Second Order Generalized Voronoi Edges which are in the boundary of the Generalized Voronoi Region, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. Note, there can be multiple Boundary Edge Fragments, but they must all come from the same Boundary Voronoi Edge.

Since $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ is a closed and connected set (actually if it is not connected, consider each connected component), the boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ can be written as the union of connected components.

$$\partial\mathcal{F}_k\big|_{\mathcal{F}_{ij}} = \bigcup_i \partial_i\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$$

where $\partial_i\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ is the i^{th} connected boundary component.

By Lemmas 9.5 and 10.2, $\mathcal{F}_{ijk}$ can only be the subset of one of the boundary components. Let $\partial_1\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ be the connected component that contains $\mathcal{F}_{ijk}$, that is, $\mathcal{F}_{ijk} \subset \partial_1\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. Furthermore, by the Extended Boundedness Assumption (Ass.

10.1) and Proposition 9.4, $\mathcal{F}_{ijk} \neq \partial_1 \mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. Therefore, $\partial_1 \mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ contains a GVG edge, GVG^2 edge(s) and perhaps Boundary Edge fragments.

$$\partial_1 \mathcal{F}_k\big|_{\mathcal{F}_{ij}} = \mathcal{F}_{ijk} \bigcup_{l_1 \in L_1} \mathcal{F}_{kl_1}\big|_{\mathcal{F}_{ij}} \left(\bigcup c_{ij}\right)$$

where $L_1 \subset L$ and $\bigcup_i L_i = L$.

For $i > 1$, $\partial_i \mathcal{F}_k\big|_{\mathcal{F}_{ij}} = \bigcup_{l_i \in L_i} \mathcal{F}_{kl_i}\big|_{\mathcal{F}_{ij}} \left(\bigcup c_{ij}\right)$. However, for $i > 2$, the existence of $\partial_i \mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ violates the Extended Boundedness Assumption by Lemma 10.3. Therefore, the Extended Boundedness Assumption implies that $\partial_1 \mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ is the only connected boundary component of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. ■

The union of the 2-Equidistant Faces is the Generalized Voronoi Diagram, and thus, this union is connected. Since all of the GVG Edges are connected (through GVG^2 edges) on each 2-Equidistant Face and all of the 2-Equidistant Faces are connected (through GVG edges), the HGVG is connected.

12 Conclusion

This paper introduced a retract-like structure called the Hierarchical Generalized Voronoi Graph. Although this structure was specifically developed for sensor based implementation, it can be used for classical motion planning as well. However, since it is defined in terms of the distance function, the HGVG readily lends itself to sensor based implementation [7]. Because of its graph-like structure, motion planning can be reduced to a 1-dimensional graph search. Simulations validating this approach for the case of 2-dimensions can be found in [5], but simulations of the 3-dimensional case are under way.

For the 3-dimensional case, it was shown that under a certain set of conditions the HGVG is connected. Proof of the higher dimensional case can be found in [6]. In the case where the Extend Boundedness Condition is not met, a linking procedure is required. That is the current area of research and will soon be included in [6].

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Appendix

A.1 Proof of Lemma 9.3

Proof: Around the interior of Generalized Voronoi Edge, $\mathcal{F}^o_{ijk}$, $d_i(x) = d_j(x) = d_k(x) < d_h(x)$ for all h and $x \in \mathcal{F}^o_{ijk}$. Let $Y = \mathcal{F}_{ij} \bigcap nhbd(x)$. By continuity of the distance function, $\forall y \in Y$, $d_i(y) = d_j(y) \leq d_k(x) < d_h(x)$. Therefore, there can not exists an $l \neq i, j, k$ for which $d_l(y) = d_k(y)$ for any $y \in Y$. That is, there can to be a $\mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}}$ which intersects the interior of the GVG edge. Continuity of the distance function only allows A GVG^2 edge can only intersect the GVG at a meet point. ▼

A.2 Proof of Lemma 9.5

Proof: $\mathcal{F}_{ijk}$ can be defined as: $\{x : \forall h \, d(h) \geq d_k(x) = d_j(x) = d_i(x)\}$. And $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ is: $\mathcal{F}_k\big|_{\mathcal{F}_{ij}} = \{x : \forall h, \, d_h(x) \geq d_k(x) \geq d_j(x) = d_i(x)\}$

Pick $x \in int(\mathcal{F}_{ijk})$. Let $Y = nbhd(x) \bigcap \mathcal{F}_{ij}$. By continuity of the distance function, for all $y \in Y$, $\forall h \, d_h(y) \geq d_k(y) \geq d_j(y) = d_i(y)$. Therefore, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ exists.

$$\begin{aligned} \mathcal{F}_{ijk} &= \{x : \forall h \, d_h(x) \geq d_i(x) = d_j(x) = d_k(x)\} \\ &\subset \partial\{x : \forall h \, d(h) \geq d_k(x) \geq d_j(x) = d_i(x)\} \\ &= \partial \mathcal{F}_k\big|_{\mathcal{F}_{ij}} \end{aligned}$$

Therefore, by definition, if $\mathcal{F}_{ijk}$ exists, then it is a subset of the boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. ▼

A.3 Proof of Lemma 9.6

Proof: Assume that $\mathcal{F}_{ijk}$ and $\mathcal{F}_{ijl}$ are on the boundary of $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$. By definition, $\forall x \in \mathcal{F}_k\big|_{\mathcal{F}_{ij}}$, $d_l(x) \geq d_k(x) \geq d_i(x) = d_j(x)$. By assumption, $\forall x \in \mathcal{F}_{ijl} \subset \mathcal{F}_k\big|_{\mathcal{F}_{ij}}, d_k(x) \leq d_l(x)$. Of course, this is the same thing as saying, $\forall x \in \mathcal{F}_{ijl} \backslash \mathcal{F}_{ijk} \subset \mathcal{F}_k\big|_{\mathcal{F}_{ij}}, d_k(x) \leq d_l(x)$. However, this is a contradiction because: $\forall x \in \mathcal{F}_{ijl} \backslash \mathcal{F}_{ijk}$, $d_l(x) < d_k(x)$. ▼

A.4 Proof of Lemma 9.7

Proof: For the sake of discussion, assume $\mathcal{F}_{ij}$ has two 3-Equidistant Faces on its boundary: $\mathcal{F}_{ijk}$ and $\mathcal{F}_{ijl}$. By definition, the existence of $\mathcal{F}_{ijk}$ and $\mathcal{F}_{ijl}$ implies $\mathcal{F}_k\big|_{\mathcal{F}_{ij}}$ and $\mathcal{F}_l\big|_{\mathcal{F}_{ij}}$, respectively exist. It can be shown that union of the Second Order Generalized Voronoi Regions on a 2-Equidistant Face is the 2-Equidistant Face, and Generalized Voronoi Regions only intersect at their boundaries. Therefore, $\mathcal{F}_k\big|_{\mathcal{F}_{ij}} \bigcap \mathcal{F}_l\big|_{\mathcal{F}_{ij}} = \{x \in \mathcal{F}_{ij} : \forall h \, d_h(x) \geq d_l(x) = d_k(x) \geq d_i(x) = d_j(x)$ which is the definition of a GVG^2 edge, $\mathcal{F}_{kl}\big|_{\mathcal{F}_{ij}}$. ▼