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A Configuration Space Friction Cone

(extended abstract)

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Abstract

This paper provides a geometric representation of friction for a rigid planar part with two translational and one rotational degrees of freedom. The construction of a generalized friction cone is accomplished by imbedding into the part's configuration space the constraints that define the classical friction cone in real space. The resulting representation provides a simple computational method for determining the possible motions of a part subjected to an applied force and torque. The representation has been used both for simulating part motions and for planning assembly operations. Generalizations to the six-dimensional configuration space of a three-dimensional part are possible.

1. Introduction

The basic problem that this paper addresses is:

> Given a rigid planar body, possibly in finite frictional contact with immovable obstacles, given an initial velocity of the body that is consistent with the contact conditions, and given an applied force and torque, what are the possible accelerations of the body?

The basic approach is to model Newton's and Coulomb's laws in configuration space [Lozano-Pérez 83]. The result is a geometric friction cone and a pair of geometric projection operators that map an initial velocity and an applied generalized force into a resulting set of possible accelerations. This set may contain zero, one, or more possible accelerations, since rigid body dynamics with Coulomb friction need not yield a unique motion. The solution proposed in this paper provides all possible accelerations consistent with the stated contact conditions and the assumptions of rigid body dynamics. We do not model deformations or impact.

2. Equations of Motion

In order to illustrate the approach, let us derive the equations of motion for a planar object in single-point contact with an immobile object. These equations will provide us with the desired representation of friction in generalized force space. We will assume that friction may be modelled as dry Coulomb friction, with friction coefficient μ.

The main difficulty in representing friction for objects with extent lies in relating forces at different points of contact. For an object with extent, frictional reaction forces at some contact points may induce torques about the object's center of mass. The two relevant issues are:

- Modelling contact friction in terms of the induced forces and torques relative to the center of mass (or center of compliance).
- Using this model to predict reaction forces and motions resulting from applied forces and torques.

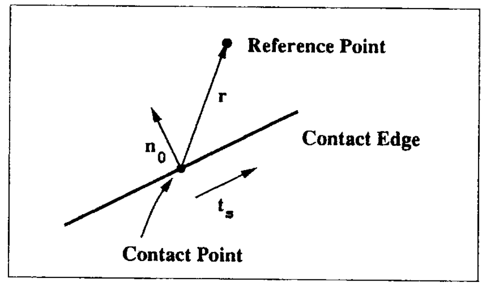

Figure 1: The local real space normal, sliding tangent, and radius vectors of a generic contact.

A rigid planar object has three degrees of freedom, two translational and one rotational. It is convenient to choose the reference point of the object at its center of mass (or more generally at its center of compliance), and to represent the object's motions in terms of generalized coordinates.

The generalized coordinates are (x, y, q), viewed as elements of the manifold $\Re^2 \times S^1_\rho$. Here S^1_ρ is the circle of radius ρ, and ρ is taken to be the radius of gyration of the object. Thus the relationship between the usual representation of orientation as an angle θ and the generalized coordinate q is $q = \rho\,\theta$.

Consider Figure 1, which depicts an abstraction of a planar object in one-point contact with some other object. Two different contacts that could give rise to this same picture are shown in Figure 2. We denote by $\mathbf{n}_0$ the unit real-space normal at the point of contact, and write $\mathbf{n}_0 = (n_x, n_y)$. We let $\mathbf{r} = (r_x, r_y)$ denote the vector from the point of contact to the moving object's reference point.

Suppose now that we permit the moving object to rotate and translate while maintaining single-point contact with the immobile object. The legal motions of the object thus constrained have two degrees of freedom. We may describe these legal motions as a two-dimensional surface in the (x, y, q) configuration space of the moving object. At any point on this surface we may construct an outward unit normal to the surface, denoted by $\mathbf{n}$.

Referring to Figure 1, we may write

$$\mathbf{n} = \frac{1}{\Delta_n}(n_x, n_y, n_q/\rho),$$

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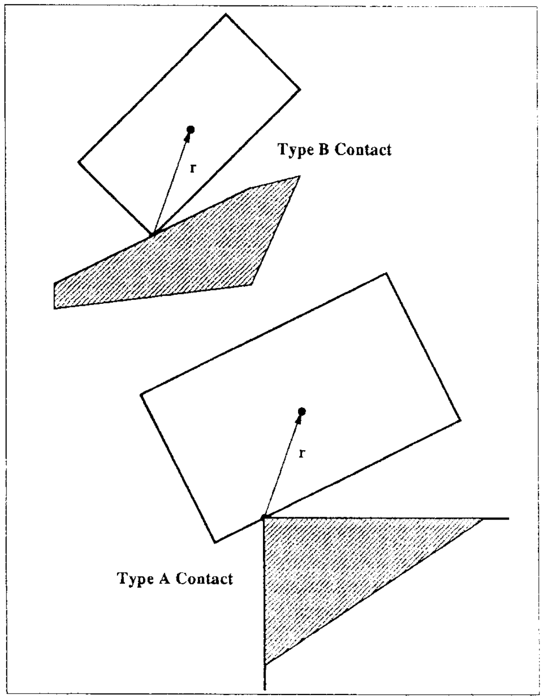

Figure 2: Two different types of contact between a moving object and an obstacle. Both examples are described by the same vectors at the point of contact, as in Figure 1. (The manner in which these vectors change, as the object moves, is different for the two types of contact.)

with $\Delta_n = (\rho^2 + n_q^2)^{1/2}/\rho$, and n_q to be determined.

It is no coincidence that the first two components of this vector are directly parallel to the real space normal $\mathbf{n}_0$. We can see that this must be so, merely from force considerations. Intuitively, one should think of the configuration space normal $\mathbf{n}$ as specifying the direction of a generalized reaction force that arises in response to a frictionless applied force acting on the surface. Consequently, since we have chosen the reference point at the center of mass, and chosen ρ to be the radius of gyration, n_q must simply be the torque induced about the reference point by a unit reaction force at the point of contact:

$$n_q = \mathbf{n}_0 \times_{2D} \mathbf{r}.$$

where $\times_{2D}$ is the two-dimensional cross product.

Just as we have modelled the normal reaction force, so too can we model the frictional reaction force. We can think of friction as acting tangentially to the physical edge of contact. Let $\mathbf{t}_s$ denote the unit tangent to the edge of contact. Then $\mathbf{t}_s$ must be of the form $\mathbf{t}_s = \pm(n_y, -n_x)$.

Friction acts along this tangent through the point of contact. For a unit frictional reaction force, the induced torque about the center of mass is therefore v_q, with

$$v_q = \mathbf{t}_s \times_{2D} \mathbf{r}.$$

Observe that $v_q = \pm(n_x r_x + n_y r_y)$.

Let us now write down the equations of motion. Figure 3 depicts a force-body diagram for the contact of Figure 1. Let

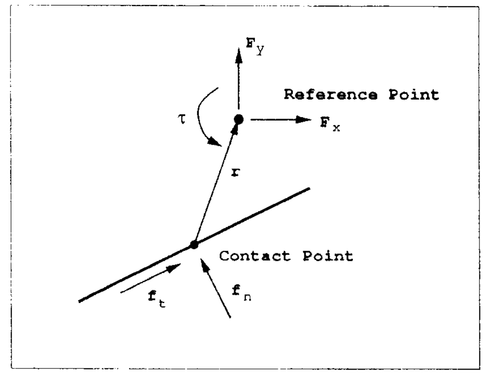

Figure 3: Applied forces at the reference point and reaction forces at the contact point for the contact of Figure 1.

$\mathbf{F}_A = (F_x, F_y, F_q)$ be a generalized applied force. In other words, the applied cartesian force is (F_x, F_y) and the applied torque is $\tau = \rho F_q$. This force is applied at the center of mass, that is, at the reference point. Let the normal reaction force at the point of contact have signed magnitude f_n, and let the frictional reaction force have signed magnitude f_t. We measure f_n along the outward normal $\mathbf{n}_0$, and f_t along the tangent vector $\mathbf{t}_s$. For the choice of $\mathbf{t}_s$ as in Figure 1, the equations of motion are: (m is the mass of the moving object)

$$\begin{aligned} f_n n_x + f_t n_y + F_x &= m a_x \\ f_n n_y - f_t n_x + F_y &= m a_y \\ f_n n_q + f_t v_q + \tau &= m \rho^2 \alpha, \end{aligned} \qquad (1)$$

with the restriction $0 \leq |f_t| \leq \mu f_n$.

Let us now view the vectors $\mathbf{n}_0$, $\mathbf{t}_s$, and $\mathbf{r}$ as elements of $\Re^3$. In other words, they are just like before, but with a third coordinate that is zero.

Then we can write the configuration space normal $\mathbf{n}$ implicitly as $\Delta_n \mathbf{n} = \mathbf{n}_0 + (\mathbf{n}_0 \times \mathbf{r})/\rho$. Clearly $\Delta_n \mathbf{n}$ describes the direction of the generalized normal reaction force.

With this fact in mind, let us define the vector $\mathbf{v}_f$ as $\mathbf{v}_f = \mathbf{t}_s + (\mathbf{t}_s \times \mathbf{r})/\rho$. In other words, $\mathbf{v}_f = \pm(n_y, -n_x, v_q/\rho)$.

We can then rewrite the equations of motion (1) in generalized coordinates as

$$f_n \Delta_n \mathbf{n} + f_t \mathbf{v}_f + \mathbf{F}_A = m\,\mathbf{a}. \qquad (2)$$

subject again to the constraint $0 \leq |f_t| \leq \mu f_n$. Here $\mathbf{a} = (a_x, a_y, \rho\alpha)$ is the part's acceleration expressed in configuration space coordinates.

3. A Generalized Friction Cone

We see now that we may construct a two-dimensional cone in generalized force space that describes the range of possible reaction forces for a given contact. Specifically, this cone describes the forces and torques acting at the object's center of mass that might arise as a result of normal and frictional reaction forces acting on the object through the contact point.

The edges of the cone are described by the two rays $\Delta_n \mathbf{n} \pm \mu\, \mathbf{v}_f$. The cone is a two-dimensional planar subset of the three-dimensional generalized force space. In order to build some intuition, let observe that the configuration space normal models

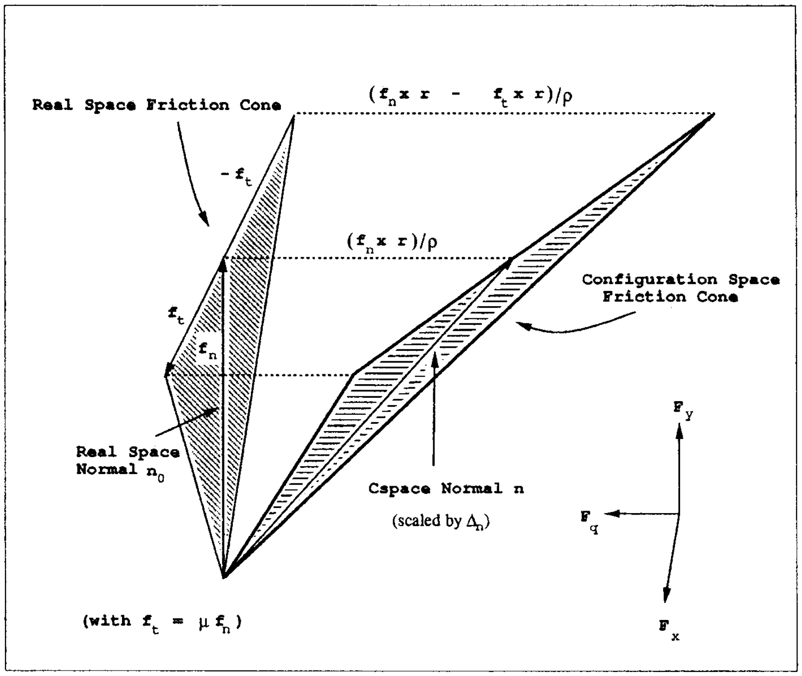

Figure 4: Relationship of real space and configuration space friction cones.

the direction of a real space normal reaction force and its induced reaction torque. This is clear from the cross-product term in the angular component n_q of the normal $\mathbf{n}$. Similarly, the edges of the friction cone model the added direction of a real space tangential reaction force and its induced torque. This is apparent from the cross-product term in the angular component of the vector $\mathbf{v}_f$. See Figure 4.

4. Ambiguities

Consider now the generalized friction cone visualized in its plane of residence. See Figure 5. The normal to this plane is given by a configuration space tangent vector that models pure rotation about the contact point. This is because pure rotations about the contact point do not require constraint forces. Let us therefore define $\mathbf{t}_r$ to be the unit tangent vector that represents pure counterclockwise rotation about the contact point. Note that $\mathbf{t}_r$ is parallel to the vector $(-r_y, r_x, \rho)$.

Define $\mathbf{t}_r^\perp$ as a unit vector that is orthogonal to both $\mathbf{n}$ and $\mathbf{t}_r$. Since there are two such vectors, we need to make some choices. Let us choose the directions of $\mathbf{t}_s$ and $\mathbf{t}_r^\perp$ so that $\mathbf{t}_r^\perp \cdot \mathbf{n}_0 \geq 0$ and $\mathbf{t}_r^\perp \cdot \mathbf{t}_s > 0$. Then $\mathbf{t}_r^\perp \cdot \mathbf{v}_f > 0$ as well. Figure 5 depicts the configuration space friction cone relative to a coordinate frame given by $\mathbf{t}_r^\perp$ and $\mathbf{n}$, where $[\mathbf{t}_r^\perp, \mathbf{n}, \mathbf{t}_r]$ form an orthonormal coordinate frame. The previous discussion implies that the generalized friction cone generally will appear asymmetric relative to the configuration space normal $\mathbf{n}$. This situation is unlike that for a real space friction cone.

One consequence of this asymmetry is that for large enough values of the coefficient of friction, the friction cone edge $\mathbf{e}^+$ actually dips below the tangent plane at the configuration space surface of contact. In other words, $\mathbf{e}^+ \cdot \mathbf{n} < 0$. This is very strange, for it says that with sufficient friction, a contact can resist an applied force pointing away from the surface of contact. In other words, were it not for friction, the applied force would actually cause the contact to break and cause the object to spin off into free space. In the full version of this paper, we show how this condition explains various motion ambiguities that are possible in the presence of friction.

5. Computing Reaction Forces

The description of the generalized friction cone as a convex subset of the plane $\mathbf{F} \cdot \mathbf{t}_r = 0$ raises the hope that we can easily compute a reaction force in a manner similar to that described in standard physics texts for blocks and point masses. Specifically, we would like to take an applied force $\mathbf{F}_A$, and simply project it tangentially onto the friction cone in order to determine a reaction force. This is essentially the correct procedure, except that we will need two projection operators, one that projects into the plane $\mathbf{F} \cdot \mathbf{t}_r = 0$, and another that projects onto the edges of the friction cone.

In this section we first consider the conditions leading to static equilibrium. Next we consider the case in which the object is initially at rest and is subjected to an applied force and torque that may cause the object to move. We show how to compute the reaction force for this case. Finally, we consider the general case in which the object is moving initially. Throughout the computation of reaction forces, we assume that the configuration space friction cone lies above the tangent plane. It is fairly straightforward to generalize the computations to the case in which the friction cone dips below the tangent plane. There are some complications, which we hinted at in Section 4. These deal with multiple or zero solutions to the equations of motion. We will not deal with these issues here, but see [Erdmann 84].

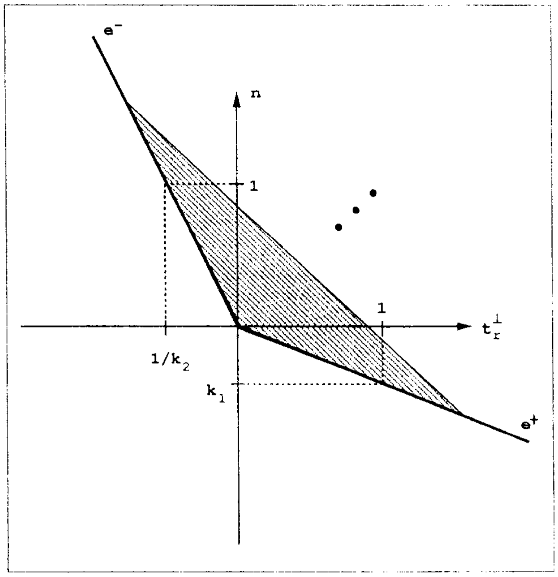

Figure 5: Planar view of the configuration space friction cone in the plane $\mathbf{F}\cdot\mathbf{t}_r = 0$. Notice that the friction cone can actually dip below the tangent plane. This is determined by the sign of the parameter k_1. (The outward normal to the configuration space surface is $\mathbf{n}$; the tangent plane is given by the vectors $\mathbf{t}_r$ and $\mathbf{t}_r^\perp$.)

5.1. Static Equilibrium

We will address the general contact case presently. In this subsection we would like to focus on one particular contact mode, namely on static equilibrium. It is natural to focus first on this contact mode, since the friction cone may be thought of as a characterization of all applied forces that can result in static equilibrium.

In order for static equilibrium to exist the following constraints must be satisfied by the applied (generalized) force $\mathbf{F}$.

(i) The normal component of the applied force at the contact point must point into the edge of contact.

(ii) The tangential component of the applied force at the point of contact must have magnitude no greater than μ times the normal component.

(iii) The tangential component of the applied force in the direction of pure rotation must be zero.

The first two constraints are the standard conditions for one-sided contact with friction. The third condition arises because the generalized friction cone is two-dimensional, whereas the generalized force space is three-dimensional. Since forces along the rotational tangent $\mathbf{t}_r$ are perpendicular to the friction cone, they cannot be opposed by any reaction force. Hence, for static equilibrium to exist, such forces must vanish.

One can use the classical friction cone constraints to translate these conditions into algebraic constraints on the applied force $\mathbf{F}$.

Specifically, we may rewrite the constraints as:

(i)' $$\mathbf{F}\cdot\mathbf{n}_0 \le 0.$$

(ii)' $$\mu\,\mathbf{F}\cdot\mathbf{n}_0 \le \mathbf{F}\cdot\mathbf{t}_s \le -\mu\,\mathbf{F}\cdot\mathbf{n}_0$$

(iii)' $$\mathbf{F}\cdot\mathbf{t}_r = 0.$$

Ideally, we would like to determine as well a similar specification of the configuration space friction cone, using the configuration space vectors $\mathbf{n}$ and $\mathbf{t}_r^\perp$. One might hope for a set of constraints of the form:

$$c_1\mathbf{F}\cdot\mathbf{n} \le \mathbf{F}\cdot\mathbf{t}_r^\perp \le c_2\mathbf{F}\cdot\mathbf{n}, \quad \text{with } \mathbf{F}\cdot\mathbf{n} \le 0. \tag{3}$$

Such a pair of constraints would select a convex subset from the plane defined by the equation $\mathbf{F}\cdot\mathbf{t}_r = 0$.

Unfortunately, as we have mentioned, the generalized friction cone can dip below the tangent plane. Thus this straightforward analogy to the real space case is not possible. Instead, we shall see in this section that the constraints (3) must be formulated as:

$$\begin{aligned} \mathbf{F}\cdot\mathbf{n} &\le k_1\mathbf{F}\cdot\mathbf{t}_r^\perp \\ \mathbf{F}\cdot\mathbf{n} &\le k_2\mathbf{F}\cdot\mathbf{t}_r^\perp \end{aligned} \tag{4}$$

with

$$\begin{aligned} k_1 &= \frac{\mathbf{t}_r^\perp\cdot\mathbf{t}_s - \mu\,\mathbf{t}_r^\perp\cdot\mathbf{n}_0}{\mu\,\mathbf{n}\cdot\mathbf{n}_0} \\ k_2 &= \frac{\mathbf{t}_r^\perp\cdot\mathbf{t}_s + \mu\,\mathbf{t}_r^\perp\cdot\mathbf{n}_0}{-\mu\,\mathbf{n}\cdot\mathbf{n}_0} \end{aligned} \tag{5}$$

Let us now derive the constraints (4) and (5) by using Coulomb's law and the conditions (i)', (ii)', and (iii)'.

Since $\mathbf{F}\cdot\mathbf{t}_r = 0$ by (iii)', and since $\mathbf{n}\cdot\mathbf{t}_s = 0$ by construction, we have that:

$$\begin{aligned} \mathbf{F}\cdot\mathbf{n}_0 &= (\mathbf{F}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{n}_0 + (\mathbf{F}\cdot\mathbf{t}_r)\mathbf{t}_r\cdot\mathbf{n}_0 + (\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{n}_0 \\ &= (\mathbf{F}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{n}_0 + (\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{n}_0 \end{aligned} \tag{6}$$

$$\begin{aligned} \mathbf{F}\cdot\mathbf{t}_s &= (\mathbf{F}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{t}_s + (\mathbf{F}\cdot\mathbf{t}_r)\mathbf{t}_r\cdot\mathbf{t}_s + (\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{t}_s \\ &= (\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{t}_s \end{aligned} \tag{7}$$

Therefore, by expanding (ii)', we get:

$$(\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{t}_s \le -\mu\left[(\mathbf{F}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{n}_0 + (\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{n}_0\right] \tag{8}$$

$$(\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{t}_s \ge \mu\left[(\mathbf{F}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{n}_0 + (\mathbf{F}\cdot\mathbf{t}_r^\perp)\mathbf{t}_r^\perp\cdot\mathbf{n}_0\right] \tag{9}$$

Equivalently,

$$(\mathbf{F}\cdot\mathbf{t}_r^\perp)\left[\mathbf{t}_r^\perp\cdot\mathbf{t}_s + \mu\,\mathbf{t}_r^\perp\cdot\mathbf{n}_0\right] \le -\mu\,(\mathbf{F}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{n}_0 \tag{10}$$

$$(\mathbf{F}\cdot\mathbf{t}_r^\perp)\left[\mathbf{t}_r^\perp\cdot\mathbf{t}_s - \mu\,\mathbf{t}_r^\perp\cdot\mathbf{n}_0\right] \ge \mu\,(\mathbf{F}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{n}_0 \tag{11}$$

The desired results (4) and (5) follow.

5.2. Computing Reaction Forces with the Object Initially at Rest

We now consider the case in which the object can move in response to an applied force. Let us assume that the object is initially at rest, and that the friction cone actually lies above the tangent plane. This means that k_1 is positive and k_2 is negative.

If we are given an applied force $\mathbf{F}_A$, we first check whether $\mathbf{F}_A\cdot\mathbf{n}$ is negative. If this is not the case, then the applied force must point either parallel to or away from the configuration space surface of contact, implying that there is no reaction force.

If $\mathbf{F}_A\cdot\mathbf{n} < 0$, then we first project $\mathbf{F}_A$ tangentially into the plane of the friction cone. Specifically, we remove the $\mathbf{t}_r$ component of the applied force, leaving the force $\mathbf{F}_A - (\mathbf{F}_A\cdot\mathbf{t}_r)\mathbf{t}_r$.

If the resulting projection points into the generalized friction cone, then it defines the reaction force. Otherwise, we must perform an additional projection, in order to project onto the edges of the friction cone. This projection is parallel to the vector $\pm\mathbf{t}_r^\perp$, again leaving the normal component of $\mathbf{F}_A$ unchanged. This ensures that the reaction force will completely cancel the normal component of the applied force, thereby preserving the contact.

Thus, if $\mathbf{F}_A$ is a generalized applied force, then the configuration space surface responds with the reaction force

$$\mathbf{F}_R = -(\mathbf{F}_A \cdot \mathbf{n})\mathbf{n} - h\,\mathbf{t}_r^\perp,$$

where h satisfies $(\mathbf{F}_A \cdot \mathbf{n})/k_1 \le h \le (\mathbf{F}_A \cdot \mathbf{n})/k_2$. If the first projection lies within the interior of the generalized friction cone, then these inequalities are strict. In that case, $h = \mathbf{F}_A \cdot \mathbf{t}_r^\perp$. Otherwise, one of the inequalities is an equality, meaning that the reaction force lies on an edge of the friction cone.

5.3. General Contact Conditions

We consider now the case in which the object may be in motion initially, while in single-point contact with some configuration space surface.

In order to maintain contact with a surface, it is necessary to restrict the range of velocities and accelerations. In particular, in order to remain on a surface of contact it must be the case that the object's velocity is tangential to the surface. This statement applies equally in real space and in configuration space. In real space, the contact velocity must be tangential to the real space edge, that is, the contact velocity normal to the edge must be zero. Similarly, in configuration space, the reference point must be moving tangentially to the configuration space surface. Its normal velocity must be zero. In short, in order to maintain contact with a surface, the following condition must hold:

$$\mathbf{v} \cdot \mathbf{n} = 0, \tag{12}$$

where $\mathbf{v}$ is the object's configuration space velocity and $\mathbf{n}$ is the configuration space normal. We write $\mathbf{v} = (v_x, v_y, \rho\,\omega)$.

The same condition does not apply to accelerations. It is not the case that either the normal contact acceleration or the normal configuration space acceleration must be constrained to be zero. This issue is related to the presence of centripetal and Coriolis terms in the general equations of motion.

The correct acceleration constraint is derived by differentiating Equation (12). If the configuration space normal $\mathbf{n}$ is constant, then the condition reduces to assuming that the normal configuration space acceleration be zero. Otherwise, an extra term appears.

5.4. General Second Variation Constraint

In general, suppose that a surface in some configuration space of n parameters $x_1, \ldots, x_n$ is represented by the implicit equation[1]

$$F(x_1, \ldots, x_n) = 0.$$

Then the first and second variation constraints are given by

$$\frac{dF}{dt} = 0 \quad \text{and} \quad \frac{d^2F}{dt^2} = 0.$$

Here t represents time. Intuitively, the constraints say that any curve on the surface given by F cannot leave that surface.

The first variation constraint reduces to:

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\frac{dx_i}{dt} = 0.$$

[1]For multiple contact there would be several such surface constraints, corresponding to the intersection of surfaces in configuration space.

This is of course just the same as the velocity constraint (12), with the configuration space normal $\mathbf{n}$ parallel to the vector $(\partial F/\partial x_1, \ldots, \partial F/\partial x_n)$.

The second variation constraint is the derivative of the first variation constraint. So

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 F}{\partial x_j \partial x_i}\frac{dx_j}{dt}\frac{dx_i}{dt} + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\frac{d^2 x_i}{dt^2} = 0. \tag{13}$$

The rightmost, single, summation is the dot product of a vector parallel to the configuration space normal with the configuration space acceleration. The leftmost, double, summation consists of centripetal and Coriolis terms.

For further details, see [Jellet 1872].

5.5. Second Variation Constraint for Type A and B Contacts

We will not derive the second variation constraints for the contacts depicted in Figure 2, merely present the results.

The top contact in that figure is known as a Type B contact (see [Lozano-Pérez 83]). It is defined by the interaction of a vertex of the moving object with an edge of an obstacle. Observe that the radius vector remains constant in length for this type of contact. The bottom contact in the figure is known as a Type A contact. It is defined by the interaction of an edge of the moving object with a vertex of an obstacle. The radius vector can now change length.

For Type B contacts the second variation constraint is of the form:

$$\mathbf{n} \cdot (\mathbf{a} + \omega^2 \mathbf{r}) = 0.$$

For Type A contacts, the second variation constraint is of the form

$$\mathbf{n} \cdot (\mathbf{a} - \omega^2 \mathbf{r} - 2\vec{\omega} \times \mathbf{v}_{xy}) = 0.$$

Here $\mathbf{v}_{xy} = (v_x, v_y, 0)$ and $\vec{\omega}$ is a vector representation of the angular velocity ω, namely as $\vec{\omega} = (0, 0, \omega)$.

We see then that the second variation constraint can be written as

$$\mathbf{n} \cdot (\mathbf{a} + \mathbf{h}) = 0,$$

for some vector $\mathbf{h}$ that is a function of the configuration space parameters and their first time derivatives.

5.6. Computing Reaction Forces

Using the second variation constraints it is possible to compute reaction forces for the general case in which the object is in single-point contact with an obstacle. The object may be moving initially, while maintaining contact with the obstacle. We simply present the results here without derivation. These results assume that the configuration space friction cone lies wholly above the tangent plane.

Let us write the reaction force as $\mathbf{F}_{\text{reaction}} = f_n\,\mathbf{n} + f_t\,\mathbf{t}_r^\perp$. Recall that this is a valid representation, since there can be no reaction force along the tangent $\mathbf{t}_r$ that represents pure rotation about the contact point. Let us also write $\mathbf{v}_f$ as $\mathbf{v}_f = v_n\,\mathbf{n} + v_t\,\mathbf{t}_r^\perp$, with $v_t > 0$. Finally, let $\mathbf{v}_0$ be the cartesian velocity of the contact point. Specifically, $\mathbf{v}_0$ is the velocity of the point on the moving object that is coincident with the contact point. Then f_n and f_t are given by the following rules:

$$f_n = \max\left\{0, \quad -(\mathbf{F}_{\text{applied}} + m\,\mathbf{h}) \cdot \mathbf{n}\right\},$$

$$C = v_t\,(\mathbf{F}_{\text{applied}} \cdot \mathbf{t}_r^\perp) - v_n\,(m\,\mathbf{h} \cdot \mathbf{n}) + \omega^2\,(m\,\mathbf{r} \cdot \mathbf{v}_f),$$

$$f_t = \begin{cases} f_n/k_2, & \text{if } \mathbf{v}_0 \cdot \mathbf{t}_s > 0, \\ f_n/k_1, & \text{if } \mathbf{v}_0 \cdot \mathbf{t}_s < 0, \\ f_n/k_2, & \text{if } \mathbf{v}_0 \cdot \mathbf{t}_s = 0 \text{ and } -C/v_t \leq f_n/k_2, \\ f_n/k_1, & \text{if } \mathbf{v}_0 \cdot \mathbf{t}_s = 0 \text{ and } -C/v_t \geq f_n/k_1, \\ -C/v_t, & \text{if } \mathbf{v}_0 \cdot \mathbf{t}_s = 0 \text{ and} f_n/k_2 \leq -C/v_t \leq f_n/k_1. \end{cases}$$

These rules generalize the projection operators of Section 5.2.

6. Multiple Contacts

It is conceptually easy to generalize this approach to multiple contacts. Specifically, for an object in multiple point contact with its environment, the net friction cone is simply the vector sum of the individual single-point friction cones. The full version of this paper generalizes the method by which contact motions may be predicted, given a set of contacts and an applied generalized force. See also [Erdmann 84].

Observe that once one can compute reaction forces for a given set of contacts, then one can solve the problem stated in the Introduction. Specifically, for any set of geometric contacts, one hypothesizes that some subset actually imparts forces to the moving object. One then computes those forces and determines the resulting motion of the object. If this motion violates no physical constraints then it is a valid solution to the equations of motion. The procedure just described requires time exponential in the number of contacts, in the worst case, since one may need to consider all possible subsets of the geometric set of contacts in order to find one or more that are physically valid. Recent results by [Baraff 90] suggest that this may be a fundamental complexity.

7. Related Work

There has been considerable work on the modelling of friction. An important contribution is the book by Jellet [Jellet 1872]. This book sets up the basic problem, modelling each contact as a defining constraint surface, then listing possible contact modes. Additionally, Jellet was well-aware of both static and dynamic ambiguities in the solution of Newton's equations with Coulomb friction. [Lötstedt 81] discusses the inconsistency and ambiguity of frictional dynamics as well, and provides a simulation-based solution. [Mason and Wang 87] further discuss this issue and provide an impact model for removing ambiguities.

Much of the work on modelling friction arose in the context of understanding the peg-in-hole problem. An important paper that motivates the work discussed in this paper is [Whitney 82]. Other important work includes [Nevins, et. al. 75], [Drake 77], [Ohwovoriole, Hill, and Roth 80], [Ohwovoriole and Roth 81], and the work by Simunovic [Simunovic 79]. More recent work with an emphasis on understanding three-dimensional peg-in-hole assemblies in the presence of friction and uncertainty includes [Caine 85] and [Sturges 88].

Of similar intent to our work is the paper by Rajan, Burridge, and Schwartz [Rajan, Burridge, and Schwartz 87]. This paper develops a characterization of the possible contact modes of a planar body in frictional contact with rigid objects. The authors split force-torque space into a number of regions, in each of which the character of the contact mode is identical.

[Brost and Mason 89] also develop a method for determining regions of invariant contact mode. Specifically, they represent both forces and contact constraints as acceleration centers in the plane. They introduce geometric operations similar to convex hull as a means for combining disparate forces. The resulting regions in the plane then fully characterize the object's contact modes.

A great advantage of both of these approaches is their two-dimensional representation of the three-dimensional force-torque space. This is achieved by noting that frictional reaction forces form infinite cones, and thus may be described as regions on the unit sphere in force-torque space.

Further details on the work discussed in this paper may be found in [Erdmann 84].

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