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Multiple-Point Contact with Friction: Computing Forces and Motions in Configuration Space

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1. Introduction

This article summarizes ideas on representing translational friction and computing reaction forces in configuration space. The configuration space of an object is the parameter space describing the object's degrees of freedom. [Lozano-Pérez 1981, 1983]. Kinematic constraint's imposed on the object by obstacles in the environment may be represented as hypersurfaces in the configuration space. Physical and dynamical constraints such as those imposed by friction may be represented as constraints in the generalized force space that accompanies the configuration space. This article develops a representation of friction in this generalized force space that is analogous to the classical friction cone in real space.

2. Problem Statement

The basic problem that this article solves is:

> Given a rigid planar body, possibly in frictional, discrete, finite, multiple-point contact with immovable rigid obstacles; given an initial velocity of the body that is consistent with the contact conditions; and given an applied force and torque; what are the possible accelerations of the body?

The basic approach is to model Newton's and Coulomb's laws in configuration space [Lozano-Pérez 1981, 1983]. The result is a geometric friction cone and a set of geometric projection operators that map an initial velocity and an applied generalized force into a resulting set of possible accelerations. This set may contain zero, one, or more possible accelerations, since rigid body dynamics with Coulomb friction need not yield a unique motion. The solution proposed in this article provides all possible accelerations consistent with the stated contact conditions and the assumptions of rigid body dynamics. We do not model deformations or impact.

This paper builds on the work discussed in [Erdmann 1991]. That earlier paper focused on single point contact. The main purpose of the current paper is to provide a method for computing reaction forces in the presence of multiple points of contact.

3. Applications, Motivation, and Related Work

3.1. Applications

Applications for the techniques discussed in this article lie in the design of parts feeders, the planning of grasping operations, the simulation of parts in contact, and the planning of error-tolerant assembly and manipulation strategies. For instance, by maintaining compliant contact between two parts it is possible to remove the uncertainty in their relative position. Deciding whether the parts can actually slide on each other during such compliant contact entails understanding the dynamics of frictional contact.

3.2. Modelling Friction

There has been considerable work on the modelling of friction. An important contribution is the book [Jellet 1872]. This book sets up the basic problem, modelling each contact as a defining constraint surface, then listing possible contact modes. Additionally, Jellet was well-aware of both static and dynamic ambiguities in the solution of Newton's equations with Coulomb friction. [Lötstedt 1981] discusses the inconsistency and ambiguity of frictional dynamics as well, and provides a simulation-based solution. [Mason and Wang 1987] further discuss this issue and provide an impact model for removing ambiguities. For work that investigates frictional ambiguities in the forward dynamics of a mechanical system see [Dupont 1992]. For an investigation of frictional ambiguities and a discussion of some unexpected behavior in the presence of infinite friction see [Lynch and Mason 1993].

Much of the work on modelling friction arose in the context of understanding the peg-in-hole problem. We have already mentioned [Whitney 1982]. Other important work includes [Nevins et al. 1975], [Drake 1977], [Ohwovoriole, Hill, and Roth 1980], [Ohwovoriole and Roth 1981], and [Simunovic 1979]. More recent work with an emphasis on understanding three-dimensional peg-in-hole assemblies in the presence of friction and uncertainty includes [Caine 1985] and [Sturges 1988]. For related work that uses three-dimensional representations of friction for testing subassembly stability, see [Boneschanscher et al. 1988].

Of similar intent to our work is the paper [Rajan, Burridge, and Schwartz 1987]. This paper develops a characterization of the possible contact modes of a planar body in frictional contact with rigid objects. The authors split force-torque space into a number of regions, in each of which the character of the contact mode is identical. [Brost and Mason 1989] also develop a method for determining regions of invariant contact mode. Specifically, they represent both forces and contact constraints as acceleration centers in the plane. They introduce geometric operations similar to convex hull as a means for combining disparate forces. The resulting regions in the plane then fully characterize the object's contact modes.

3.3. Pushing

The problem considered in this article is fully planar. In other words, all motions and forces lie in the plane. Another class of manipulation problems is nearly planar. For instance, consider an object resting on a horizontal table under the influence of gravity. If one applies forces parallel to the table, then the motion of the object will be planar. However, the magnitude of the frictional reaction force between the object and the table will be determined by a non-planar quantity, namely by the normal force perpendicular to the table. We do not consider such problems in this article. However, these problems of planar pushing and sliding have been studied extensively. See, in particular, [Mason 1986], [Mani and Wilson 1985], [Peshkin 1986], [Peshkin and Sanderson 1986], [Brost 1988], and [Goyal, Ruina, and Papadopoulos 1991]. More recent work on analyzing and synthesizing pushing motions given multiple-point contact is presented in [Lynch 1992], and work on planning pushing strategies is presented in [Akella and Mason 1992].

3.4. Grasping

An important application for which an understanding of friction is required is the stable grasping of parts. Given a geometric description of a part and a manipulator, along with coefficients of friction, the problem is one of synthesizing stable grasps and regrasps in order to manipulate the part purposefully. Some important work in this area may be found in the papers [Fearing 1984], [Cutkosky 1985], [Mishra, Schwartz, and Sharir 1987], [Nguyen 1988], and [Trinkle 1989].

3.5. Parts Orienting

[Erdmann and Mason 1988] discuss an application relevant to the design of feeder systems and the orienting of parts. The authors developed a planner based on the frictional representation to be discussed in the current article.

3.6. Impact

Although impact is beyond the scope of this article, the modelling of friction leads naturally to a discussion of impact. Thus a full simulation or planning system must be able to take account of both impact and friction. The literature on impact is vast and the problem is still not well understood. We mention here a few of the papers related to impact, friction, and inconsistency. We

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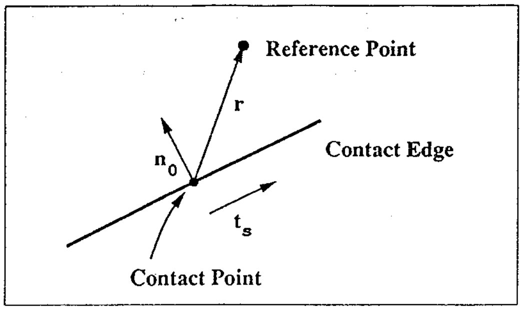

Figure 1: The local real space normal, sliding tangent, and radius vectors of a generic contact.

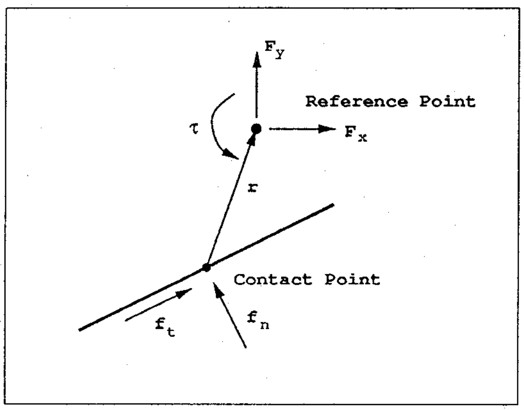

Figure 2: Applied forces at the reference point and reaction forces at the contact point.

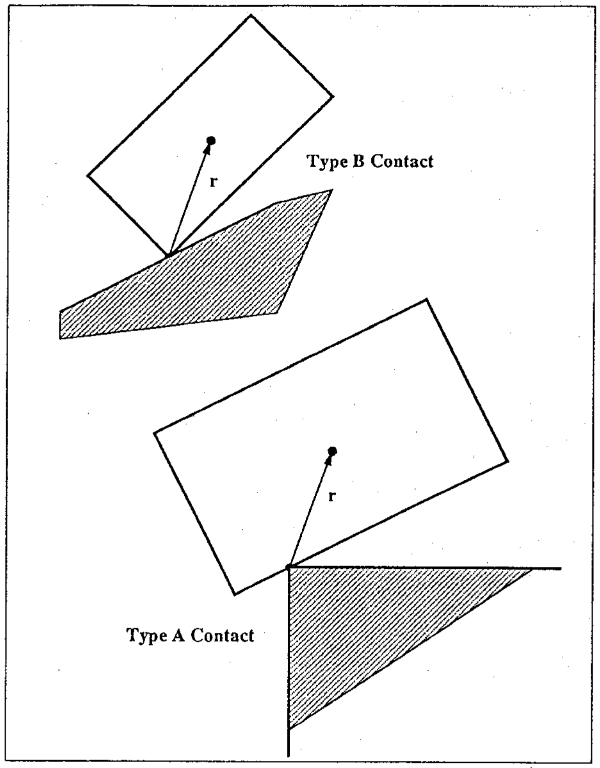

Figure 3: Two different types of contact between a moving object and an obstacle. Both examples are described by the same vectors at the point of contact, as in Figure 1. (The manner in which these vectors change, as the object moves, is different for the two types of contact.)

have already mentioned that [Mason and Wang 1987] propose an impact model for removing frictional inconsistency. For a thorough investigation of impact along this line of attack see also [Wang and Mason 1987] and [Wang 1989]. On the subject of inconsistencies in impact models without friction see [Featherstone 1986]. Finally, for recent work that addresses impact inconsistency in the presence of friction, see [Wang, Kumar, and Abel 1992].

4. Single-Point Contact Friction Cone

4.1. Contacts in Configuration Space

This article builds on the representation of friction for single-point contact developed in [Erdmann 1991]. We assume that the reader is familiar with that development or the one contained in [Erdmann 1984]. We briefly recall here the notation of that work.

We assume that friction may be modelled as dry Coulomb friction, with friction coefficient μ.

A rigid planar object has three degrees of freedom, two translational and one rotational. It is convenient to choose the reference point of the object at its center of mass, and to represent the object's motions in terms of generalized coordinates.

The generalized coordinates are (x, y, q), viewed as elements of the manifold $\Re^2 \times S^1_\rho$. Here S^1_ρ is the circle of radius ρ, and ρ is taken to be the radius of gyration of the object. Thus the relationship between the usual representation of orientation as an angle θ and the generalized coordinate q is $q = \rho\,\theta$.

Consider Figure 1, which depicts an abstraction of a planar object in one-point contact with some other object. Two different contacts that could give rise to this same picture are shown in Figure 3. Additionally, Figure 2 depicts a force diagram for the same contact. We denote by $\mathbf{n}_0$ the unit real-space normal at the point of contact, and write $\mathbf{n}_0 = (n_x, n_y)$. We let $\mathbf{r} = (r_x, r_y)$ denote the vector from the point of contact to the moving object's reference point. We will often think of these vectors as vectors in configuration space. This simply entails adding a third component that is zero.

Suppose now that we permit the moving object to rotate and translate while maintaining single-point contact with the immobile object. The legal motions of the object thus constrained have two degrees of freedom. We may describe these legal motions as a two-dimensional surface in the (x, y, q) configuration space of the moving object. At any point on this surface we may construct an outward unit normal to the surface, denoted by $\mathbf{n}$.

Referring to Figure 1, we may write

$$\mathbf{n} = \frac{1}{\Delta_n}(n_x, n_y, n_q/\rho),$$

with $\Delta_n = (\rho^2 + n_q^2)^{1/2}/\rho$, and $n_q = \mathbf{n}_0 \times_{2D} \mathbf{r}$, where $\times_{2D}$ is the two-dimensional cross product.

Just as we have modelled the normal reaction force, so too can we model the frictional reaction force. We can think of friction as acting tangentially to the physical edge of contact. Let $\mathbf{t}_s$ denote the unit tangent to the edge of contact. Then $\mathbf{t}_s$ must be of the form $\mathbf{t}_s = \pm(n_y, -n_x)$. Friction acts along this tangent through the point of contact. For a unit frictional reaction force, the induced torque about the center of mass is therefore v_q, with

$$v_q = \mathbf{t}_s \times_{2D} \mathbf{r}.$$

4.2. A Vector Decomposition

The development of [Erdmann 1991] leads to the following vector decomposition.

$$\begin{aligned}
\mathbf{r} &= (r_x, r_y, 0) \\
\mathbf{n}_0 &= (n_x, n_y, 0) \\
\mathbf{n} &= \frac{\rho}{\sqrt{\rho^2 + n_q^2}}(n_x, n_y, n_q/\rho) \\
\mathbf{t}_r &= \frac{1}{\sqrt{r_x^2 + r_y^2 + \rho^2}}(-r_y, r_x, \rho) \\
\mathbf{t}_r^{\perp} &= \pm\frac{1}{\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2}}(\delta_1, -\delta_2, \delta_3) \\
\mathbf{t}_s &= \pm(n_y, -n_x, 0), \\
\mathbf{v}_f &= \pm(n_y, -n_x, v_q/\rho)
\end{aligned} \qquad (1)$$

where

$$n_q = n_x r_y - n_y r_x$$

$$
\begin{aligned}
v_q &= n_x r_x + n_y r_y \\
\delta_1 &= \rho^2 n_y - n_q r_x \\
\delta_2 &= \rho^2 n_x + n_q r_y \\
\delta_3 &= \rho r_y n_y + \rho r_x n_x
\end{aligned} \tag{2}
$$

Here $\mathbf{t}_r$ is the tangent vector in configuration space that represents pure rotation about the contact point. The vector $\mathbf{t}_r^\perp$ is perpendicular to both $\mathbf{n}$ and $\mathbf{t}_r$. The vector $\mathbf{t}_s$ represents pure sliding at the contact point. Finally, the vector $\mathbf{v}_f$ describes the frictional reaction force in configuration space.

It is convenient to choose the directions of the vectors $\mathbf{t}_r^\perp$, $\mathbf{t}_s$, and $\mathbf{v}_f$ so that $\mathbf{t}_r^\perp \cdot \mathbf{n}_0 \geq 0$, $\mathbf{t}_r^\perp \cdot \mathbf{t}_s > 0$, and $\mathbf{v}_f \cdot \mathbf{t}_s > 0$.

4.3. A Configuration Space Friction Cone

We see that we may construct a two-dimensional cone in generalized force space that describes the range of possible reaction forces for a given contact. Specifically, this cone describes the forces and torques acting at the object's center of mass that might arise as a result of normal and frictional reaction forces acting on the object through the contact point. The edges of the cone are described by the two rays $\Delta_n \mathbf{n} \pm \mu \mathbf{v}_f$. The cone is a two-dimensional planar subset of the three-dimensional generalized force space. See Figure 4.

4.4. Computing Reaction Forces

The brief version of the procedure for computing reaction forces for single-point contact goes as follows. If we are given an applied force $\mathbf{F}_A$, we first check whether $\mathbf{F}_A \cdot \mathbf{n}$ is negative. If this is not the case, then the applied force must point either parallel to or away from the configuration space surface of contact, implying that there is no reaction force.

If $\mathbf{F}_A \cdot \mathbf{n} < 0$, then we first project $\mathbf{F}_A$ tangentially into the plane of the friction cone. Specifically, we remove the $\mathbf{t}_r$ component of the applied force, leaving the force $\mathbf{F}_A - (\mathbf{F}_A \cdot \mathbf{t}_r)\mathbf{t}_r$.

If the resulting projection points into the generalized friction cone, then it defines the reaction force. Otherwise, we must perform an additional projection, in order to project onto an edge of the friction cone. This projection is parallel to the vector $\pm \mathbf{t}_r^\perp$, again leaving the normal component of $\mathbf{F}_A$ unchanged. This ensures that the reaction force will completely cancel the normal component of the applied force, thereby preserving the contact.

Thus, if $\mathbf{F}_A$ is a generalized applied force, then the configuration space surface responds with the reaction force

$$\mathbf{F}_R = -(\mathbf{F}_A \cdot \mathbf{n})\mathbf{n} - h\,\mathbf{t}_r^\perp,$$

where h is chosen so that $\mathbf{F}_R$ lies within the friction cone. If the first projection lies within the interior of the generalized friction cone, then $h = \mathbf{F}_A \cdot \mathbf{t}_r^\perp$. Otherwise, h is chosen so $\mathbf{F}_R$ lies on the appropriate edge of the friction cone.

The procedure just outlined does not hold in all cases. It applies when the object is initially at rest and the configuration space friction cone lies above the tangent plane. For a more detailed analysis the reader is referred to [Erdmann 1984] and [Erdmann 1991].

5. Multiple Contacts

For an object in multiple-point contact with its environment, the net friction cone is simply the vector sum of the individual single-point friction cones. This section elucidates this assertion and generalizes the method by which contact motions may be predicted, given a set of contacts and an applied generalized force.

With these tools in hand, one can solve the basic problem stated in Section 2. Specifically, for any set of geometric contacts, one hypothesizes that some subset of these contacts actually imparts forces to the moving object. One then computes those forces and determines the resulting motion of the object. If this motion violates no physical constraints then it is a valid solution to the equations of motion. The procedure just described requires time exponential in the number of contacts, in the worst case, since one may need to consider all possible subsets of the geometric set of contacts in order to find one or more that are physically valid. Recent results by [Baraff 1990] suggest that this may be a fundamental complexity.

5.1. Equations of Motion on the Intersection of Surfaces

Multiple contact in real space corresponds to the intersection of several hyper-surfaces in configuration space. Consider a part in multiple-point contact with an immobile environment. Each point of contact defines a hyper-surface in the part's configuration space. The hyper-surface represents the constraints on the part's motion imposed by that contact. Several points of contact impose several constraints. The constraints are satisfied along the "surface" given by the intersection of all the individual hyper-surfaces.

Each hyper-surface possesses a configuration space normal and configuration space tangents, following the usual vector decomposition for a single-point contact as given in Section 4.2. Suppose that there are k contact points. We will designate the various vectors by using the notation of Section 4.2 and by subscripting. For instance, let $\mathbf{n}_{0i} = (n_{xi}, n_{yi}, 0)$ be the real space normal at contact point i, and let $\mathbf{r}_i$ be the radius vector from the i^{th} point of contact to the reference point. Let n_{qi} be the cross product $\mathbf{n}_{0i} \times_{2D} \mathbf{r}_i$ and let $\mathbf{n}_i$ be the configuration space normal for the i^{th} point of contact. And let v_{qi} be $\mathbf{t}_{si} \times_{2D} \mathbf{r}_i$, as in the definition of the vector $\mathbf{v}_{fi}$ that is used to define the edges of the i^{th} friction cone.

Finally, let the real space reaction force at point i be given by the scalar f_{ni} along the normal direction, and by the scalar f_{ti} along the tangential direction. The equations of motion are then simply

$$
\begin{aligned}
F_x + \sum_{i=1}^{k} f_{ni}\, n_{xi} + \sum_{i=1}^{k} f_{ti}\, n_{yi} &= m\, a_x \\
F_y + \sum_{i=1}^{k} f_{ni}\, n_{yi} - \sum_{i=1}^{k} f_{ti}\, n_{xi} &= m\, a_y \\
\tau + \sum_{i=1}^{k} f_{ni}\, n_{qi} + \sum_{i=1}^{k} f_{ti}\, v_{qi} &= m\, \rho^2\, \alpha,
\end{aligned} \tag{3}
$$

where

$$0 \leq |f_{ti}| \leq \mu f_{ni}, \quad \text{for every } i = 1, \ldots, k.$$

From the form of these equations it is apparent that the possible range of reaction forces is the vector sum of the range of reaction forces due to each individual point contact. This is just the principle of superposition.

5.2. Three-Point Contact Example

Consider the example of Figure 5. The object shown is in three-point contact with a horizontal edge. All contacts are Type B contacts. Since the configuration space normals span a two-dimensional subspace of generalized force space, the object is not overconstrained. The object can slide horizontally, while maintaining all points of contact. Additionally, by breaking contact at the other points, the object can rotate about either of the extreme contact points.

In this section we derive the composite friction cone for this object.

5.2.1. The Vector Decomposition

All vectors will be subscripted by the number of the contact point. The left contact point is point number 1; the right, number 2; and the middle, number 3.

All three points have the same real space normal, namely $\mathbf{n}_0 = (0, 1, 0)$. Given the conventions of Section 4.2, the sliding tangents are:

$$
\begin{aligned}
\mathbf{t}_{s1} &= (1, 0, 0) \\
\mathbf{t}_{s2} &= (-1, 0, 0) \\
\mathbf{t}_{s3} &= (1, 0, 0)
\end{aligned}
$$

The radius vectors are

$$
\begin{aligned}
\mathbf{r}_1 &= (1, 1, 0) \\
\mathbf{r}_2 &= (-1, 1, 0) \\
\mathbf{r}_3 &= (0, 1, 0)
\end{aligned} \tag{4}
$$

Consequently, the configuration space normals and the vectors that define the friction cone are given by the formulas

$$
\begin{aligned}
\Delta_1 \mathbf{n}_1 &= (0, 1, -1/\rho) \\
\Delta_2 \mathbf{n}_2 &= (0, 1, 1/\rho) \\
\Delta_3 \mathbf{n}_3 &= (0, 1, 0) \\
\mathbf{v}_{f1} &= (1, 0, 1/\rho)
\end{aligned}
$$

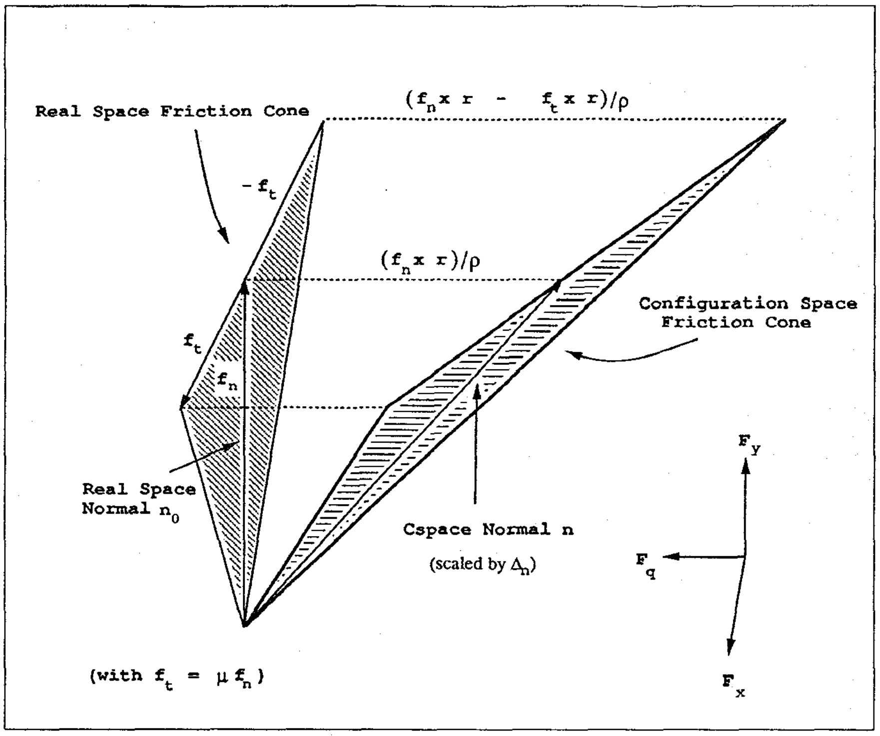

Figure 4: Relationship of real space and configuration space friction cones.

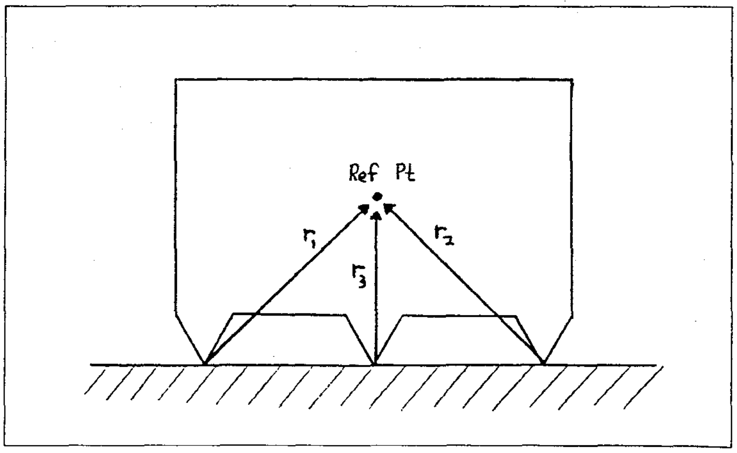

Figure 5: Three-point contact. Two principal motions involve sliding on the surface and rotating at the extreme contact points.

$$v_{f2} = (-1, 0, -1/\rho)$$
$$v_{f3} = (1, 0, 1/\rho)$$

5.2.2. The Friction Cone

Recall that the edges of the configuration space friction cone arising from a point of contact are given by the two vectors $\Delta_n \mathbf{n} \pm \mu \mathbf{v}_f$. Thus the individual friction cones for the three points of contact listed above are described by the following edge vectors:

Contact Point 1:	$(\mu, 1, (\mu - 1)/\rho)$	$(-\mu, 1, -(\mu + 1)/\rho)$
Contact Point 2:	$(\mu, 1, (\mu + 1)/\rho)$	$(-\mu, 1, -(\mu - 1)/\rho)$
Contact Point 3:	$(\mu, 1, \mu/\rho)$	$(-\mu, 1, -\mu/\rho)$

The friction cones are drawn in Figure 6, using $\rho = 1$ and $\mu = 0.25$. The composite friction cone is simply the volume between these three friction cones. In the example, the edges of the middle friction cone lie in the planes formed by the edges of the exterior friction cones. In general, this need not be the case. See [Erdmann 1984] for an example.

5.3. Classes of Contacts

A multiple-point contact defines a collection of possible contact modes. In particular, a point moving on the intersection of several configuration space surfaces can certainly remain on that intersection. That is one contact mode. However, the point can also move to a higher dimensional intersection, by leaving one or more of the surfaces. Each such intersection defines yet another contact mode. In the previous example, the object can slide, thereby maintaining all three points of contact, or it can rotate, thereby breaking two of the points of contact, or it can move into free space, thereby breaking all contacts.

This observation suggests that in analyzing a multiple-point contact, it is also necessary to analyze all smaller subclasses of contacts. In particular, given an applied force, it may be necessary to compute possible motions for all possible subsets of contacts. In the absence of friction, for a generic contact only one of these motions will be physically feasible, meaning that

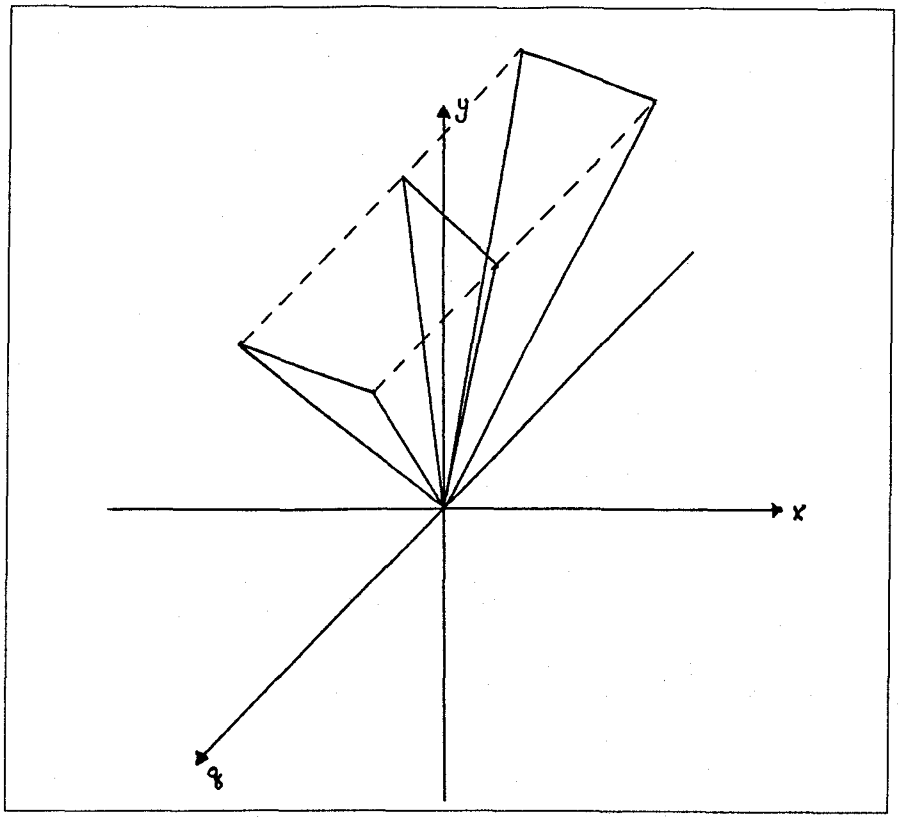

Figure 6: Friction cone for the example of Figure 5. The three individual friction cones are described by the triangular sheets. The composite friction cone comprises the volume between the individual friction cones, as indicated by the dashed lines. The cones are all of infinite extent in the $+y$-direction.

only one such motion will violate no physical constraints.[1] In the presence of friction, however, there may exist several motions consistent with the physical contacts and a given applied force. Indeed, we know that frictional indeterminacies arise even for some single-point contacts. For multiple-point contact, acceleration magnitudes may be ambiguous as well, possibly taking on any of a continuous set of values. Detailed examples are contained in [Erdmann 1984].

In the previous three-point contact, let us outline the possible motions. The details are left to the reader, and are worked out for a general contact in the next several sections. The general approach would entail first enumerating all possible geometric contact modes, then retaining only those consistent with the resulting equations of motion. We will here simply jump to the conclusion of such an enumeration.

If the coefficient of friction is small, then the fact that the middle configuration space friction cone is the sum of the exterior friction cones, and the fact that for small μ the cones all lie above their respective tangent planes, implies that the motion resulting from a given applied force will be unique. In particular, a force along the horizontal sliding direction will cause sliding motion. One can see this by inspecting the composite friction cone. A force with no torque component must project onto one of the side faces of the composite friction cone, that is, onto a face formed by two distinct friction cones. We will see in Section 5.5 that this projection must be parallel to the sliding tangent, and thus the resulting motion is pure sliding. In order to verify that this is the only physically feasible motion, imagine projecting the pure force onto one of the individual friction cones using the method outlined in Section 4.4. The resulting net force would induce an acceleration that rotates about one of the contacts, while trying to push through the table at the other contacts. This is physically impossible. Consequently, single-point contact cannot occur. Several points of contact must be involved in determining the net motion of the object.

In contrast, the application of a generalized force dominated by a large torque component must cause a rotation about one of the extreme contact points. The applied force must project onto one of the exterior one-point contact friction cones. Neither of the other two friction cones is involved in the computation of the reaction force, indicating that the effective contact involves a single point.

[1] Ambiguities may exist for non-generic contacts, e.g., a vertex touching another vertex.

5.4. Predicting Reaction Forces

Suppose that an object is in k-point contact with its (immobile) environment, and suppose that a generalized applied force is acting on the object. This section develops a method for predicting the possible motions of the object.

5.4.1. Consistency of Second Variation Constraints

The motion of the object must satisfy the first and second variation constraints for each configuration space hyper-surface at which there is contact. The first variation constraint says simply that the contact velocity must be tangential to any surface of contact that is maintained and away from the surface for any contact that is broken. We are here only interested in those contacts for which the contact velocity is tangential to the surface of contact. If the velocity is away from the surface, we may as well assume that there is no reaction force at that point of contact, and therefore we may ignore the contact. If the velocity is into the surface, then a collision must result. Collisions are beyond the scope of this article.

The key constraints are therefore the second variation constraints. See [Erdmann 1991] for a derivation of the second variation constraints. We may write the second variation constraints in the form:

$$\mathbf{a} \cdot \mathbf{n}_i \geq -h_i, \quad i = 1, \ldots, k. \tag{5}$$

Here $\mathbf{a}$ is the configuration space acceleration of the object, and $\mathbf{n}_i$ is the outward-pointing configuration space normal at contact i. Each h_i is some appropriate function of configuration and generalized velocity that encodes dynamic forces such as centripetal and Coriolis forces [Erdmann 1991].

The second variation constraints are constraints on normal acceleration. We have written the constraints as inequalities. This is because not all of the k points of contact need to be maintained. For those contacts that are maintained the constraints must be equalities, while for those that are broken the inequalities ensure that the motion of the object at the contacts does not violate any physical barriers.

5.4.2. Case Analysis for Computing Reaction Forces

The method for computing reaction forces is a slightly involved case analysis. Here is the outline of this procedure:

- Given a set of k contacts:

1. Check whether zero reaction force is possible.
2. Check whether zero net force is possible.
3. Consider all subsets of m contacts, for $m = 1, \ldots, k$. For each subset determine a reaction force. In computing this reaction force, use a procedure that depends on the dimension of the normal space spanned by the m normals. There are three possibilities:
   (a) The dimension of the normal space is 1.
   (b) The dimension of the normal space is 2.
   (c) The dimension of the normal space is 3.

   Once a reaction force has been computed, check that it satisfies all k second variation constraints[2] given by (5).

We now examine the steps of this procedure. In addition, Section 5.5 contains a fairly detailed description of Step 3.(b).

5.4.3. Computing Possible Motions: The Cases of Zero Reaction Force and Zero Net Force

Given k contacts we first examine two extreme scenarios, then look at all subsets of contacts. The two extreme scenarios are: (1) No reaction force, and (2) a reaction force that completely cancels the applied force.

Checking the possibility of zero reaction force is easy. One merely checks that the acceleration resulting from Newton's equations in free space violates no physical constraints, that is, that it satisfies all k second variation constraints.

In order to check whether a canceling reaction force is possible, one must ascertain whether the negative applied force lies in the convex of possible reaction forces. First, suppose the object is initially at rest. In this case one must compute the convex sum of all possible one-point contact friction cones, then check whether the negative of the applied force lies within this convex. If so, then a reaction force exists that cancels the applied force. Since the object is initially at rest, it will remain at rest.

More generally, suppose the object has non-zero initial velocity. We wish to determine whether there exists a reaction force that completely cancels the applied force and satisfies the second variation constraints (5). It is easy to check the second variation constraints. If a canceling reaction force exists then acceleration is zero, so one must simply verify that $h_i \geq 0$, for all $i = 1, \ldots, k$. If these conditions are not satisfied then a canceling reaction force cannot exist.

Suppose that the conditions are satisfied. Contact can be maintained instantaneously at all contacts for which $h_i = 0$. Assume without loss of generality that this occurs for $i = 1, \ldots, m$, for some m. As in the case of zero velocity, one must ascertain whether the negative applied force lies in the convex of possible reaction forces, now formed only by the m contacts at which $h_i = 0$.

There is one additional wrinkle. Coulomb's law implies that at any contact where there is relative motion, the reaction force must lie on an edge of the friction cone, namely on the edge that opposes the direction of motion. Thus one uses only those edges to build the convex used in the test. If the negative of the applied force lies in the resulting convex, then there exists a reaction force that cancels the applied force, while instantaneously maintaining contact at the m contacts.

5.4.4. Computing Possible Motions: The General Cases

Having discussed these two extreme cases, we next turn our attention to the intermediate cases. In order to compute possible reaction forces, one must first find all subsets of the given k contacts that can satisfy the second variation constraints with equality. For each such subset one then computes a possible reaction force. If the resulting net acceleration violates no physical constraints at the other points of contact, then it is a possible motion of the object. In order to test whether the net acceleration violates any physical constraints one simply plugs the acceleration into all k second variation constraints. Some of these constraints will be satisfied with equality, by construction. The remaining inequality constraints are the true test conditions. For instance, in the example of Figure 5 the test that arises frequently is whether a rotation about some point of contact has the correct sign, that is, whether the object rotates away from the other points of contact.

Suppose we are given a set of m contact points that satisfy the second variation constraints with equality (with $m \leq k$). Let us denote the m configuration space normal vectors at these contacts by $\mathbf{n}_1, \ldots \mathbf{n}_m$. There are three primary cases to consider, determined by the dimensionality of the space spanned by these normals.

5.4.5. The Case of a One-Dimensional Normal Space

Suppose the dimension of the span of the m configuration space normals is one. If m is one, then we compute a reaction force using the method outlined in Section 4.4. If m is greater than one, then all m configuration space normals are parallel. This is an interesting but highly singular situation. We will not consider it in this article. Let us simply mention that the key to analyzing redundant contacts in which the normal space has dimension one is to determine the motion directions that maintain two or more of the contacts. There will be a finite number of such motion directions. For each such direction the object effectively has only a single degree of freedom, even though the dimension of the normal space is one. This simplifies the analysis considerably. There are two approaches. One is to write out the equations of motion explicitly using the fact that the motion direction is known. The other is to treat the contact as a singular instance of a contact state in which the normal space really has dimension two. For the latter approach the analysis of Step 3.(b) of the procedure outlined in Section 5.4.2 applies.

5.4.6. The Case of a Three-Dimensional Normal Space

Suppose the dimension of the span of the m configuration space normals is three. Then either the object does not move in response to the applied force, or the object must break contact and reduce the dimensionality of the normal space. Since we already test whether the contacts can fully balance the applied force, and since we handle all subclasses of contacts, there is nothing further to discuss.

5.4.7. The Case of a Two-Dimensional Normal Space

Suppose the dimension of the span of the m configuration space normals is two. This is the most interesting case. If planar objects could exist, they would frequently find themselves in this situation. For instance, two point contact is a generic stable situation in a planar (x, y) world in which gravity acts along some direction in the plane.

Since we have assumed that the second variation constraints are equalities at the m contacts, we can write these constraints as:

$$\begin{pmatrix} n_{x1} & n_{y1} & n_{q1}/\rho \\ n_{x2} & n_{y2} & n_{q2}/\rho \\ \vdots & \vdots & \vdots \\ n_{xm} & n_{ym} & n_{qm}/\rho \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ \rho\alpha \end{pmatrix} = - \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix}.$$

Now consider the system of equations

$$\begin{pmatrix} n_{x1} & n_{y1} & n_{q1}/\rho \\ n_{x2} & n_{y2} & n_{q2}/\rho \\ \vdots & \vdots & \vdots \\ n_{xm} & n_{ym} & n_{qm}/\rho \end{pmatrix} \begin{pmatrix} h_x \\ h_y \\ h_q \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix}. \qquad (6)$$

Denote by $\mathbf{h}'$ a solution to this system. By assumption there is at least one such solution. Different solutions differ from each other by vectors that are perpendicular to all the configuration space normals $\mathbf{n}_i$, $i = 1, \ldots, m$.

The solution $\mathbf{h}'$ is a fictitious generalized acceleration, with corresponding fictitious generalized force $m\mathbf{h}'$. Suppose the object is in some contact state with known velocities. Given an applied force, we wish to compute a reaction force so that the resulting net acceleration satisfies the second variation constraints. One easy way to do this is to add to the given applied force our fictitious force $m\mathbf{h}'$. The resulting second variation constraints on acceleration then take the intuitively desirable form that the acceleration normal to the constraints be zero.

Formally, suppose $\mathbf{F}_A = (F_x, F_y, \tau/\rho)$ is the applied generalized force. Consider the effective applied force $\mathbf{F}'_A = \mathbf{F}_A + m\mathbf{h}'$. Similarly, define $\mathbf{a}' = \mathbf{a} + \mathbf{h}$. Finally, let $\mathbf{F}_R$ be the (yet to be computed) reaction force. We can then rewrite the system of equations

$$\mathbf{F}_A + \mathbf{F}_R = m\mathbf{a}$$

as

$$\mathbf{F}'_A + \mathbf{F}_R = m\mathbf{a}'.$$

Consequently, relative to the effective applied force $\mathbf{F}'_A$, the second variation constraints become $\mathbf{n}_i \cdot \mathbf{a}' = 0$, $i = 1, \ldots, m$.

Suppose for a moment that there is no friction. Then we compute a reaction force $\mathbf{F}_R$ by perpendicularly projecting the negative effective applied force $-\mathbf{F}'_A$ onto the normal space of the contact set. The resulting projection may or may not lie in the positive span of the normals $\mathbf{n}_1, \ldots, \mathbf{n}_m$. If so, then it defines the reaction force. Otherwise, there is no valid reaction force. This means that it is impossible to maintain this particular set of m contacts given the applied force $\mathbf{F}_A$. Any legal reaction force must come from some other contact set.

Observe that the particular choice of $\mathbf{h}'$ used in this computation is irrelevant, so long as it satisfies Equations (6). This is because different solutions $\mathbf{h}'$ differ by vectors perpendicular to the normal space, that is, by vectors along which the projection occurs.

[2]For some non-generic contacts, the conjunction (5) may be relaxed, thus possibly increasing motion ambiguity.

Now, let us return to the case in which there is friction. In order to compute a reaction force, we perform the same projection operation as before. This is forced upon us by the second variation constraints. However, we do not require that the resulting projected vector lie in the positive span of the normals, or even in the normal space. Instead, we require only that the projected vector must lie in a particular subset of the composite friction cone. Indeed, any projected vector that defines a reaction force consistent with Coulomb's law is acceptable. In Section 5.5 we examine this consistency requirement and the entire projection operation in more detail.

5.5. Details of the Projection Algorithm for a 2D Normal Space

This section discusses in detail the algorithm for computing reaction forces given a contact set whose normal space has dimension two.

Suppose there are k geometric contacts. Suppose further that the span of the configuration space normals at these contacts has dimension two. Of these k contacts we consider a subset of m contacts whose span also has dimension two. (The more general case in which the span of the k normals has dimension three while the span of the m normals has dimension two is an easy extension of the case we discuss here.)

We assume, as in Section 5.4, that the m contacts satisfy the second variation constraints with equality, and we assume further that the remaining $k - m$ contacts satisfy the second variation constraints with inequality. In the remainder of this section we analyze the possible reaction forces obtained from this subset of m contacts. Any resulting legal reaction force will result in a motion that satisfies all k second variation constraints, some with equality, some with inequality.

5.5.1. Form of the Reaction Force

Let $\mathbf{t}_c$ be the configuration space vector that is perpendicular to the normal space spanned by the configuration space normals $\mathbf{n}_1, \ldots \mathbf{n}_k$. We note in passing that $\mathbf{t}_c$ is a common configuration space tangent vector for all points of contact. It corresponds differentially to the one-degree-of-freedom motion that the object can perform while retaining contact at all m points of contact. In principle, for finite motions, some of these contacts may be broken. However, we can determine this ahead of time, for instance by looking at the second variation constraints, and thus we may assume without loss of generality that all m contacts may be maintained.

Following the reasoning and notation of Section 5.4.7, we know that any valid reaction force $\mathbf{F}_R$ arising from an applied force $\mathbf{F}_A$ must be of the form

$$\mathbf{F}_R = -\mathbf{F}_A - m\mathbf{h}' + c\,\mathbf{t}_c, \tag{7}$$

where $\mathbf{h}'$ is any solution to the system of equations (6) and c is some real number.

Our goal in the remainder of the article is to determine the legal values of c. This is most easily done by noting that any reaction force must be a combination of friction cone edges. In particular, a reaction force that is interior to a single-point friction cone may be viewed as the sum of two forces parallel to the two friction cone edges. Depending on the value of tangential contact velocities and accelerations, a given friction cone edge may or may not be able to participate in forming a reaction force. Thus, by looking at the contact velocities and accelerations, we can constrain the legal values of c. As we have mentioned, the resulting set may contain zero, one, or more legal values of c.

5.5.2. Legal Friction Cone Edges

Let $\mathbf{r}_i$ be the radius vector from contact point i to the center of mass. We can obtain the tangential contact velocities and accelerations from the configuration space velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ by the formulas:

$$\begin{aligned} \mathbf{v}_{0i} \cdot \mathbf{t}_{si} &= \mathbf{v} \cdot \mathbf{v}_{fi}, \\ \mathbf{a}_{0i} \cdot \mathbf{t}_{si} &= \left(\mathbf{a} + \omega^2 \mathbf{r}_i\right) \cdot \mathbf{v}_{fi}. \end{aligned} \tag{8}$$

Recall that $\mathbf{t}_{si}$ is the sliding tangent vector at contact i. And $\mathbf{v}_{fi}$ is the configuration space vector that, together with the configuration space normal $\mathbf{n}_i$, defines the friction cone edges for contact i. Also, $\mathbf{v}_{0i}$ is the contact velocity and $\mathbf{a}_{0i}$ is the contact acceleration of the object at contact i, while ω is the object's angular velocity.

By analogy to the one-point analysis, a reaction force at a contact point must lie on one or the other of the friction cone edges if the sliding contact velocity is non-zero. When the sliding contact velocity is zero, then the sliding contact acceleration determines the friction cone edge. If the sliding contact acceleration is also zero, then the reaction force can be in the interior of the friction cone.

Suppose the reaction force is $\mathbf{F}_R$. In order to compute the tangential acceleration at the point i, Equation (8) implies that one must consider the net force plus the term $m\,\omega^2\mathbf{r}_i$. Consequently, let us define for each contact point a variant of net force, call it $\mathbf{F}_i$:

Contact Conditions			c-Validity Intervals	
$\mathbf{v} \cdot \mathbf{v}_{fi}$	$\mathbf{t}_c \cdot \mathbf{v}_{fi}$	$\mathbf{F}_i \cdot \mathbf{v}_{fi}$	I_i^+	I_i^-
> 0	—	—	$\emptyset$	$[-\infty, +\infty]$
< 0	—	—	$[-\infty, +\infty]$	$\emptyset$
$= 0$	> 0	—	$[-\infty, c_i]$	$[c_i, +\infty]$
$= 0$	< 0	—	$[c_i, +\infty]$	$[-\infty, c_i]$
$= 0$	$= 0$	> 0	$\emptyset$	$[-\infty, +\infty]$
$= 0$	$= 0$	< 0	$[-\infty, +\infty]$	$\emptyset$
$= 0$	$= 0$	$= 0$	$[-\infty, +\infty]$	$[-\infty, +\infty]$

Table 1: Friction cone validity intervals as functions of the contact velocity and acceleration. A "—" means that the condition is irrelevant. The configuration space velocity of the object is given by $\mathbf{v}$. See also Equations (9) and (10), and see the text for a description of the other terms.

$$\mathbf{F}_i = \mathbf{F}_A + \mathbf{F}_R + m\,\omega^2\mathbf{r}_i.$$

Plugging in Equation (7), we get

$$\mathbf{F}_i = -m\mathbf{h}' + m\,\omega^2\mathbf{r}_i + c\,\mathbf{t}_c. \tag{9}$$

$\mathbf{F}_i$ effectively determines the acceleration of the object at the i^{th} contact point.

When the contact velocity is zero, then the sign of $\mathbf{F}_i \cdot \mathbf{v}_{fi}$ determines on which edge of the friction cone the reaction force must lie. Both $\mathbf{F}_R$ and $\mathbf{F}_i$ are functions of c. Thus to each friction cone edge one can assign a validity interval in c space. If c lies in this interval, then the edge can participate in forming the reaction force determined by c. Otherwise, it can not.

Let I_i^+ be the interval corresponding to the "positive" friction cone edge, that is, to the edge $\Delta_i\mathbf{n}_i + \mu\mathbf{v}_{fi}$. To say that $c \in I_i^+$ is to say that the edge $\Delta_i\mathbf{n}_i + \mu\mathbf{v}_{fi}$ may be used to compute the reaction force $\mathbf{F}_R$ given by Equation (7). Similarly, let I_i^- be the interval corresponding to the "negative" friction cone edge $\Delta_i\mathbf{n}_i - \mu\mathbf{v}_{fi}$.

Let $\mathbf{v}$ be the configuration space velocity of the reference point. Then the validity intervals are given by Table 1. In the table, the numbers $\{c_i\}$, where defined, are given by the following formula:

$$c_i = \frac{m\left(\mathbf{h}' - \omega^2\mathbf{r}_i\right) \cdot \mathbf{v}_{fi}}{\mathbf{t}_c \cdot \mathbf{v}_{fi}}. \tag{10}$$

[Also note that $\mathbf{F}_i \cdot \mathbf{v}_{fi}$ is independent of c whenever $\mathbf{t}_c \cdot \mathbf{v}_{fi} = 0$.]

5.5.3. An Algorithm for Projecting onto the Composite Friction Cone

The previous analysis provides an algorithm for computing reaction forces. The reaction force $\mathbf{F}_R(c) = -\mathbf{F}_A + m\mathbf{h}' + c\,\mathbf{t}_c$ is a function of the parameter c. Corresponding to the m points of contact there are $2m$ friction cone edges. To each edge e we associate a validity interval I_e, computed using Table 1. Relative to the assumption that contact forces arise precisely from the given m contact points, $\mathbf{F}_R(c)$ is a valid reaction force if, and only if, it can be written as a linear combination of friction cone edges $e_1, \ldots, e_j$, with c in the intersection of all the validity intervals $I_{e_1}, \ldots, I_{e_j}$.

The algorithm operates as follows. First, we sort the numbers $\{c_i\}$, along with $+\infty$ and $-\infty$. This splits c space into a number of intervals, and a number of points (the endpoints of the intervals). For instance:

$$-\infty < c_1 < \ldots < c_m < +\infty.$$

We can associate with any such interval, and with any such endpoint, a set of friction cone edges. This set consists of those friction cone edges whose validity intervals contain the given c space interval or endpoint. For instance, we would associate with $[c_i, c_{i+1}]$ all the edges $\{e_j\}$ whose validity intervals $\{I_{e_j}\}$ satisfy $[c_i, c_{i+1}] \subseteq I_{e_j}$.

For each interval $[c_i, c_{i+1}]$ (and endpoint c_i), we then determine the values of c for which $\mathbf{F}_R(c)$ can be written as a positive linear combination of all the associated friction cone edges $\{e_j\}$. This is a simple matter of intersecting the line segment $\{-\mathbf{F}_A - m\mathbf{h}' + c\,\mathbf{t}_c \mid c \in [c_i, c_{i+1}]\}$ with the semi-infinite cone defined by the positive span of the edges $\{e_j\}$. If this intersection is non-empty, then it determines a set of legal reaction forces $\mathbf{F}_R(c)$.

6. Summary

This article has developed a generalized friction cone for representing friction in configuration space. This configuration space friction cone models both the reaction forces and the reaction torques generated by a point of contact. The friction cone thus represents the range of generalized reaction forces that may be generated by a point of contact. For multiple points of contact, the composite configuration space friction cone is simply the vector sum of the individual one-point contact friction cones.

The article presented a method for computing reaction forces, given a contact state and an applied force. In the case of single-point contact this method involves two projection operators. The first operator projects the negative applied force into the plane of the friction cone. The second operator projects onto the friction cone itself. In the case of multiple-point contact, the method involves enumerating possible contacts. An important case is given by a set of contacts whose configuration space normals span a two-dimensional space. For each such two-dimensional contact set the object has one degree of remaining motion freedom. Given an applied force, one computes a reaction force with the aid of a single projection operator. This operator projects the negative applied force onto the composite friction cone by projecting along a line in generalized force space. The line is parallel to the tangent that describes the object's remaining degree of freedom.

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