On the inconsistency of rigid-body frictional planar mechanics

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Abstract

This paper reviews the problem of a thin rigid rod sliding on a horizontal surface in the plane, which is commonly cited as an example of the inconsistency of planar rigid-body Newtonian mechanics. We demonstrate the existence of a consistent solution, using Routh's analysis of rigid-body impact.

1. Introduction

It is widely accepted in the mechanics literature that Newtonian mechanics of rigid planar bodies with Coulomb friction is sometimes inconsistent, that is, that problems can be posed that have no solution satisfying the axioms of the theory in question. Excellent treatments of this issue are given by Lötstedt (1981), and Erdmann (1984), who independently constructed the *sliding rod problem* to demonstrate the inconsistency, and by Kilmister and Reeve (1966) who treat the issues of uniquess and existence in a more general context. Lötstedt attributes his example to Béghin (1923) and Klein (1909), and cites Painlevé (1895) for the first examples of this kind. A variation of the sliding rod problem provides an example of ambiguities, i.e. problems with more than one solution satisfying the axioms. (See Rajan et al. 1987, and the works cited above.)

The consistency of rigid-body mechanics is an interesting issue in the abstract, but also has some important practical ramifications. Robotics research, in particular, is exploring automatic systems for solving a number of problems in mechanics, such as testing whether a structure of objects is stable, where to place kinematic constraints (such as fingertips) so as to move, or immobilize, an object, and computer simulations for off-line programming and debugging of robotic manipulator systems. Since rigid-body mechanics provides the underlying model for some of these systems, inconsistencies and ambiguities in rigid-body mechanics are of practical importance. For example, it is awkward if a robot simulation encounters a situation with no possible outcome. Taylor et al. (1987) describe a similar difficulty in the context of planning.

The existence of inconsistencies in rigid-body mechanics is sometimes hard to accept. For the reader who has not faced this issue before, we offer the following intuitive explanation, which may make the possibility of inconsistencies more plausible, if not more palatable. To begin, consider the problem of a system of a finite number of particles subject to Newton's laws, and suppose, for concreteness, that the only forces acting among the particles are the result of mutual gravitation. Now, for any given state, i.e. a specification of the instantaneous positions and velocities of the particles, a total force acting on each particle is uniquely determined. The change in state is obtained by integrating these forces, and is likewise well-defined and uniquely determined.

The laws of rigid-body mechanics with Coulomb friction are not as simple as the laws of the system described above. In particular, Coulomb's law of sliding friction does not specify the force as a unique function of the state. Rather, it imposes *constraints* on the force, the state, and the change of state. The law does not suggest an effective means of determining the contact forces, and, in some cases, search is required to find a set of contact forces satisfying the constraints. Given this state of affairs, it is not too surprising to find that the search might turn up more than one solution (ambiguity) or fail to turn up any solutions (inconsistency).

This argument suggests the plausibility of inconsistency, but to prove inconsistency is a different matter. It suffices to demonstrate a problem with no solution satisfying the axioms of the theory. The example in question is a thin rigid rod sliding along a horizontal surface. In nice cases, the contact produces a force that balances the gravitational force, so that the end of the rod either continues sideways, or accelerates away from the surface. However, with a particular choice of the dynamic parameters, we can arrange for all feasible finite contact forces to generate an angular acceleration of the rod that accelerates the end of the rod into the surface, rather than away from the surface. Section 2 of the paper develops this analysis in detail.

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In section 3 the problem is resolved by recognizing that we have an *impact* problem, even though the rod is initially moving along, and not into, the surface. It is possible to construct impulsive forces, fully in accord with the relevant axioms, that do not accelerate the end of the rod downwards. Small impulsive forces are subject to the same constraint as finite forces, but a large enough impulse can instantaneously halt the rod's sideways motion, after which the constraint imposed by Coulomb on additional impulse is considerably relaxed. The details originate in Routh's (1860) treatment of rigid-body impact, which is further developed by Keller (1986) and Wang and Mason (1987).

2. Finite force analysis of the sliding rod problem

In this section we recapitulate Lötstedt's analysis of the sliding rod problem, introducing geometrical methods where Lötstedt relies primarily on algebraic methods. Lötstedt's methods are more suitable than ours when generality is important, but we believe that the geometrical methods are easier to understand.

Consider the rigid rod and horizontal surface of Figure 1. Following Lötstedt, we assume that the mass is distributed symmetrically about the geometrical midpoint of the rod. The ends of the rod are at distance l from the center of gravity.

We have a coordinate system aligned with the surface, and let (x, y) denote the location of the center of mass, (x_c, y_c) the lower end of the rod, θ the angle of the rod with respect to a horizontal reference, (f_{cx}, f_{cy}) the contact force acting on the rod, (f_x, f_y) an applied force acting at the center of mass, and τ an applied torque. The mass of the rod is m, the angular inertia J, and the coefficient of friction at the contact is a constant μ, whether sliding occurs or not.

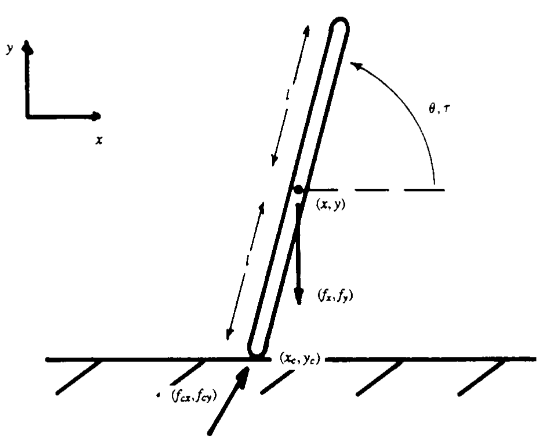

Figure 1: A rigid rod on a frictional surface.

Purely kinematic considerations provide the following relations for the position, velocity, and acceleration of the end of the rod:

$$x_c = x - l\cos\theta \quad (1)$$

$$y_c = y - l\sin\theta \quad (2)$$

$$\dot{x}_c = \dot{x} + \dot{\theta} l \sin\theta \quad (3)$$

$$\dot{y}_c = \dot{y} - \dot{\theta} l \cos\theta \quad (4)$$

$$\ddot{x}_c = \ddot{x} + \ddot{\theta} l \sin\theta + \dot{\theta}^2 l \cos\theta \quad (5)$$

$$\ddot{y}_c = \ddot{y} - \ddot{\theta} l \cos\theta + \dot{\theta}^2 l \sin\theta \quad (6)$$

For simplicity, we will assume the mass and the angular inertia to be 1. (The reader should not assume a uniform distribution of mass, but might imagine a weightless rigid rod with two point masses at unit distance from the center of mass.) Newton's laws give the following equations of motion:

$$\ddot{x} = f_x + f_{cx} \quad (7)$$

$$\ddot{y} = f_y + f_{cy} \quad (8)$$

$$\ddot{\theta} = \tau + l f_{cx} \sin\theta - l f_{cy} \cos\theta \quad (9)$$

We can obtain the equations of motion expressed with respect to the coordinates of the contact point, by combining equations 5–9:

$$\ddot{x}_c = f_x + \tau l \sin\theta + f_{cx}\left(1 + l^2 \sin^2\theta\right) + f_{cy}\left(-l^2 \cos\theta \sin\theta\right) + \dot{\theta}^2 l \cos\theta \quad (10)$$

$$\ddot{y}_c = f_y - \tau l \cos\theta + f_{cy}\left(1 + l^2 \cos^2\theta\right) + f_{cx}\left(-l^2 \cos\theta \sin\theta\right) + \dot{\theta}^2 l \cos\theta \quad (11)$$

To produce the inconsistency, we stipulate an initial translational motion to the left, with a gravitational applied force:

$$\dot{x}_c < 0 \quad (12)$$

$$\dot{y}_c = 0 \quad (13)$$

$$\dot{\theta} = 0 \quad (14)$$

$$f_x = 0 \quad (15)$$

$$f_y = -g \quad (16)$$

$$\tau = 0 \quad (17)$$

Also substituting

$$f_{cx} = \mu f_{cy} \quad (18)$$

and simplifying, we obtain

$$\ddot{y}_c = -g + a f_{cy} \quad (19)$$

where

$$a = \frac{l^2}{2}\left(\frac{l^2 + 2}{l^2} + \cos 2\theta - \mu \sin 2\theta\right) \quad (20)$$

The contact conditions dictate that both $\ddot{y}_c$ and f_{cy} be non-negative, which, from inspection of equation 19 implies that a must be positive. To complete the example, then, we choose values for l, μ, and θ to obtain a negative a. In particular, $\mu = \tan 30°$, $\theta = 75°$, and $l = 4$ render a negative.

We can best explain the situation in terms of the *instantaneous acceleration center* (Hall 1961). First, consider a motionless rod subjected to a rotational acceleration about a center (x_a, y_a), as in Figure 2. The acceleration at each point is perpendicular to a line drawn from the acceleration center, and proportional in magnitude to the length of that line. It is apparent that any point to the left of the acceleration center, including the bottom of the rod, will have a downwards component of acceleration.

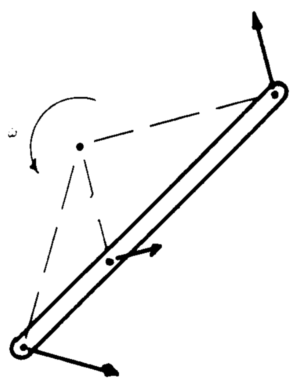

Figure 2: If the contact point were to the left of the instantaneous acceleration center, the rod would penetrate the surface.

Now, to apply this observation to the sliding rod problem, consider Figure 3. Note first that the rod is subject to two forces, one of which, the gravitational force, is fixed. The contact force is constrained by Coulomb's law to lie on a ray, making an angle $\tan^{-1}\mu$ from the contact normal. Its magnitude is unconstrained. The contact force and the gravitational force always intersect at the same point, so we can express the total force as a single force acting on a line passing through that intersection point. The family of feasible (finite) forces is illustrated in the figure. Now, for each feasible force, we can plot an instantaneous acceleration center. All of the acceleration centers fall on a single horizontal ray, which is delimited on the left by a line perpendicular to the contact force and passing through the center of mass. The distance from the ray to the center of mass varies as ρ^2. (ρ is defined to be the radius of gyration, i.e. $\sqrt{\frac{I}{m}}$.) Now, by decreasing ρ (or, equivalently, leaving ρ at a constant 1 and increasing l) it is easy to see that we can keep the feasible acceleration centers to the right of the bottom end of the rod, which, as we observed earlier, implies that the bottom of the rod accelerates into the table.

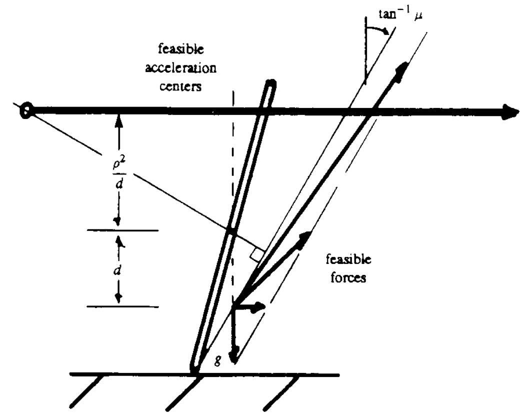

Figure 3: The locus of feasible acceleration centers. An inconsistency occurs when the locus lies entirely to the right of the contact point.

3. Impact analysis of the sliding rod problem.

The resolution of the sliding rod problem lies in viewing the rod/table interaction as a collision, involving impulsive forces. It may seem paradoxical to have a collision between two objects that are not approaching one another (and even, as we shall see, between two objects that are touching but moving away from one another!) but such a collision is surely preferable to an inconsistency in our theory. And, in any case, we have no choice, the collision being admitted by the theory.

In this section, we follow Wang and Mason (1987), which is based on Routh's (1860) analysis. We keep most of the previous section's notation, with some changes to simplify the analysis. The contact point is instantaneously at the origin, and we use, for example, v_{cx} to indicate a velocity rather than our earlier $\dot{x}_c$. P_n and P_t denote the normal and tangential components of impulse, respectively. We do not include any other applied forces. Any finite forces would be negligible relative to the impact forces, although the existence of an external force, such as the gravitational force in the last section, can determine whether the impact *must* occur, or *might* occur.

The following kinematic relations must hold:

$$v_{cx} = v_x + y\omega \quad (21)$$

$$v_{cy} = v_x - x\omega \quad (22)$$

$$\Delta v_{cx} = \Delta v_x + y\Delta\omega \quad (23)$$

$$\Delta v_{cy} = \Delta v_y - x\Delta\omega \quad (24)$$

where $\Delta v_{cx} = v_{cx} - v_{cx0}$ etc., and the following impulse-momentum laws relate the effect of impulse:

$$
\begin{aligned}
m\Delta v_x &= P_t & (25)\\
m\Delta v_y &= P_n & (26)\\
m\rho^2\Delta\omega &= P_t y - P_n x & (27)
\end{aligned}
$$

Substituting into the kinematic equations, we obtain:

$$
\begin{aligned}
\Delta v_{cx} &= \frac{P_t}{m} + y\frac{P_t y - P_n x}{m\rho^2} & (28)\\
\Delta v_{cy} &= \frac{P_n}{m} - x\frac{P_t y - P_n x}{m\rho^2} & (29)
\end{aligned}
$$

Now, the reason that impact works is that we can obtain enough zorch to instantaneously cancel the tangential motion at the contact point. We will call the condition of zero tangential relative motion *sticking*. We also define a condition called *maximum compression*, occuring at zero normal relative motion. Each of these conditions defines a linear relation between P_n and P_t. To find the sticking condition, we set $v_{cx} = 0$, obtaining:

$$0 = v_{cx0} + P_t\frac{\rho^2 + y^2}{m\rho^2} - P_n\frac{xy}{m\rho^2} \qquad (30)$$

To find the maximum compression condition, we set $v_{cy} = 0$, obtaining:

$$0 = v_{cy0} + P_n\frac{\rho^2 + x^2}{m\rho^2} - P_t\frac{xy}{m\rho^2} \qquad (31)$$

These two linear relations define lines in *impulse space*, which are plotted in Figure 4, using the same parameter values as the previous section. We have also plotted the line $P_t = \mu P_n$ through the origin, making an angle $\tan^{-1}\mu$ with the vertical. Using these three lines we can construct an impulse that satisfies the laws of Newton and Coulomb and preserves the rigidity of the rod. Although the impact is assumed to be instantaneous, it is convenient to think about the impulse accumulating from zero. Using Figure 4, the *characteristic point* representing the cumulative impulse begins at the origin, and moves along the line $P_t = \mu P_n$. The reason is that differential impulse is force, and since the rod is sliding leftwards, the differential force must obey Coulomb's law. Note, however, that eventually the characteristic point reaches the line of sticking. For an impulse on the line of sticking, the tangential velocity has halted, leaving greater freedom in the possible additional differential impulse. Now Coulomb's law allows $dP_t \leq \mu dP_n$, resisting any impending resumption of sliding. In this case, the characteristic point can satisfy this constraint by moving along the sticking line.

To complete the construction of the total impulse, we push the characteristic point to the line of maximum compression. A perfectly plastic collision would terminate at this point, with the rod's bottom instantaneously at rest with respect to the surface. A perfectly elastic collision would continue until the normal component of impulse is doubled, and would pop away from the surface, with the path of the rod's bottom end perpendicular at the surface. Intermediate cases, with coefficients of restitution between 0 and 1, terminate between these two extremes, and bounce away from the surface with varying amounts of energy.

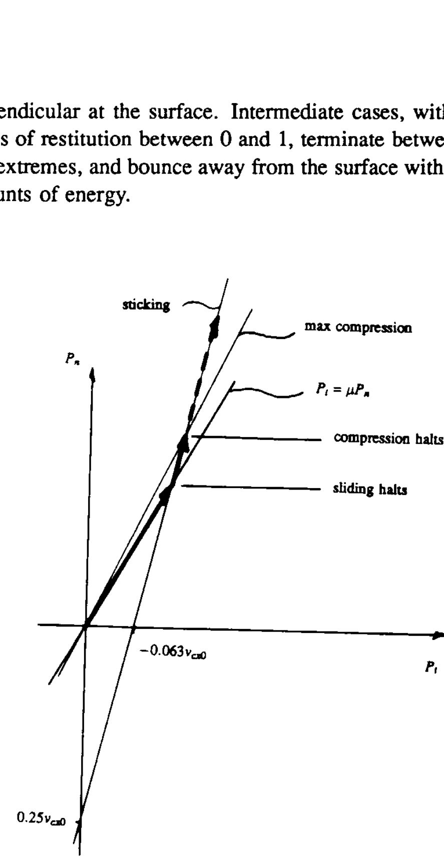

Figure 4: Impact analysis. An impulse is constructed that preserves the rigidity of the rod and surface, and satisfies Coulomb's law.

4. Remarks

We have already observed that the sliding rod problem can be varied to demonstrate ambiguities, as well as inconsistencies. The existence of an impact solution resolves the inconsistencies, raising the number of solutions from zero to one. But for problems already having a solution, the existence of an impact solution increases the ambiguity. For example, if we set $g = 0$, the rod can skim along the surface with zero contact force, or the impact can occur. Kilmister and Reeve (1966, p. 79), adopt a *principle of constraints* that resolves the additional ambiguity:

> constraints shall be maintained by forces, so long as this is possible; otherwise, and only otherwise, by impulses.

However, the basis for this principle is unclear.

Given a solution of a well-known example of inconsistency in rigid-body mechanics, the next question is whether other examples have been demonstrated. We know of one other example of inconsistency, the "second-order impact" problem reproduced from (Featherstone 1986) in Figure 5. A block is sliding in a frictionless channel of the same dimension, closed off at one end by a smooth curve. The impulse must be parallel to the change in velocity, i.e. orthogonal to the contact normals. For frictionless contacts, this is an inconsistency.

We note in parting that the sliding rod problem has important ramifications in the analysis of impact, besides its obvious relevance to the foundations of rigid-body mechanics. In our earlier work, (Wang and Mason 1987) we neglected the possibility of zero approach velocity, and we should also note that the impact might occur for small negative approach velocities. As far as we know there is no previous treatment of these possibilities.

The sliding rod problem also presents difficulties in the existing definitions of the coefficient of restitution. Newton's definition of the coefficient of restitution is defined as the ratio of the initial and final normal velocities, and seems to admit only the perfectly plastic solution in this case. Poisson's definition of the coefficient of restitution, which relates the normal impulses during compression and restitution, is applicable, but its application is not well-defined in problems for which two distinct phases of compression and restitution might not be present, such as an impact involving negative approach velocities.

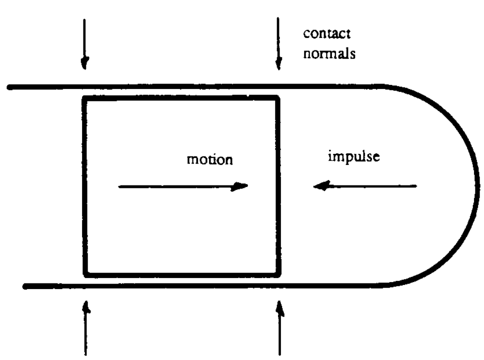

Figure 5: Second-order impact (Featherstone 1986). A change in velocity is required, in a direction orthogonal to all contact normals. For frictionless contact, this results in an inconsistency.

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