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Two Graphical Methods for Planar Contact Problems

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Abstract

Graphical methods are often applied to planar mechanics problems, especially in the context of robotic manipulation, but there are limitations. In particular, the *friction cone* is an elegant representation of the set of forces generated by a single frictional contact, but there was previously no simple extension to problems of *multiple* frictional contacts. This paper shows a simple generalization of the friction cone to include multiple contacts. The paper demonstrates application to dynamics problems and manipulation planning problems.

1 Introduction

This paper describes a graphical representation of force, called the "moment-labeling". The moment labeling representation is useful in analyzing the dynamics of a planar rigid body in contact with fixed bodies, with Coulomb friction. It is also useful in planning robot manipulator programs. This paper presents the moment-labeling representation, shows how to analyze planar frictional contact problems, and demonstrates some simple examples of motion synthesis.

The moment-labeling representation is related to an earlier graphical representation of force, described by (Brost and Mason 1989), called the "dual representation". The dual representation is somewhat more flexible, but less easily understood, than the moment-labeling representation. Both methods are closely related to the use of screw coordinates, and particularly to the techniques developed by Erdmann (1984) and by Rajan *et al.* (1987). In fact, the moment-labeling representation and the dual representation can be viewed as special cases, restricted to planar problems, but considerably simpler than the more general method. The graphical methods are ideal for humans, whereas the more general methods are better suited to a computer implementation.

2 Moment-labeling

Suppose we have some set of forces F that is closed under positive linear combination. (By "force", we mean "force and torque".) That is, for any two forces f_1, f_2 in F, $k_1 f_1 + k_2 f_2$ is also in F, for all non-negative k_1, k_2. Now, if we restrict our attention to the plane, there is a particularly simple way of representing and manipulating these sets of forces. Let F_+ be the set of all points x in the plane such that every force in F produces a non-negative moment at x; i.e.

$$F_+ = \{x \mid \text{for every } f \in F,\ x \times f \geq 0\}.$$

F_- is defined similarly:

$$F_- = \{x \mid \text{for every } f \in F,\ x \times f \leq 0\}.$$

Another way of saying this is that points in the plane are labeled. If all the forces make positive (or zero) moments, label the point $+$. If all the forces make negative (or zero) moments, label the point $-$. The points labelled $+$ form the F_+ region, and the points labelled $-$ form the F_- region. F_+ and F_- are convex regions in the plane, and they share points only on their boundaries—their interiors do not overlap—except that in the trivial case $F = \{(0,0,0)\}$, F_+ and F_- each consist of the entire plane.

We can use the regions F_+ and F_- to represent the original set of forces F. Consider some force f, and its line of action. Because f has a positive or zero moment at every point in F_+, the line of action must pass F_+ on the right. The line of force may touch the boundary, but may not pass through the interior. Similarly, the line of force must pass F_- on the left. Every force satisfying these constraints is in F, and every force in F satisfies these constraints, so (F_+, F_-) perfectly represents F. Some simple cases are shown in Figure 1.

Manipulation of force sets is very simple with the moment-labeling representation. Two operations are very common: the convex hull $\mathrm{CH}(F, G)$ of two sets, to determine the set of possible resultants of two sets of forces; and the intersection $F \cap G$ of two sets. These two operations are very straightforward under the moment-labeling representation.

- Convex hull of two force sets is implemented by intersection of the moment-labelings:

$$\mathrm{CH}(F, G) \Rightarrow (F_+ \cap G_+, F_- \cap G_-)$$

- Intersection of two force sets is implemented by convex hull of the moment-labelings:

$$F \cap G \Rightarrow (\mathrm{CH}(F_+, G_+), \mathrm{CH}(F_-, G_-))$$

The power of the representation is best demonstrated by the ease with which several friction cones may be combined, to describe the possible resultants of multiple frictional contacts

(Figure 2). We construct + and − regions for each individual friction cone, then intersect the +'s and the −'s. The resultant construction generalizes the conventional friction cone to multiple frictional contacts.

The next section shows how to apply the moment-labeling representation to dynamics problems.

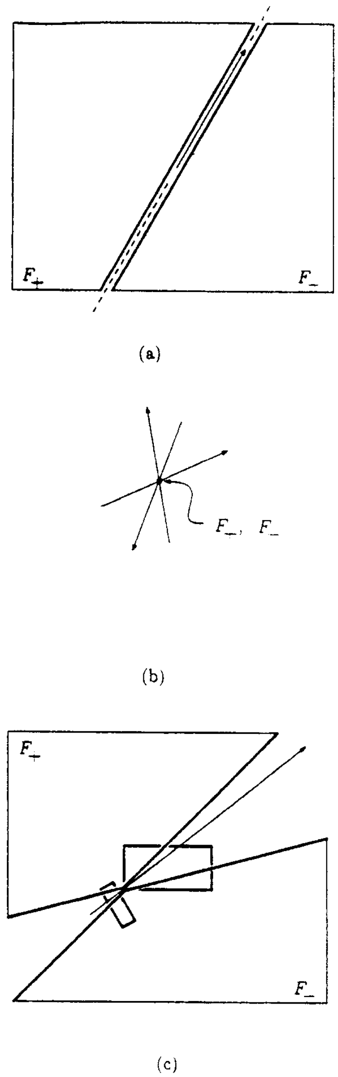

Figure 1: Some examples of the moment-labeling representation of force. (a) A line of force (i.e. all forces sharing the same line of action) divides the plane: points to the right of the line of force are in F_-, points to the left are in F_+, and points on the line of force are in both sets. (b) The set of all forces passing through a common point x yields $F_+ = F_- = \{x\}$. (c) A friction cone consists of all the forces that can be applied through a point contact obeying Coulomb's law. It is represented as shown, with the apex of the cone falling in both sets.

2.1 Graphical solution of the planar contact problem

Using the moment-labeling representation of force sets, we derive a graphical solution of the planar contact problem. The underlying approach is similar to that described by Erdmann (1984) and Rajan *et al.* (1987). We assume a mobile rigid polygon in contact with fixed polygons in the plane. The problem is to construct the mapping between applied force and object acceleration. We describe the approach in full, and describe two examples: sliding a block along a wall, and two-point insertion of a peg in a hole.

The first step in solving a planar contact problem is to enumerate the *contact modes*: for each contact, whether the relative motion is left-sliding, right-sliding, rolling, or parting. Since there are four choices at each contact, it would seem that as many as 4^n combinations might occur, for n contacts. Fortunately, only a few of these ($O(n^2)$) are kinematically consistent. These can be enumerated by an extension of Reuleaux' (1876) method for analyzing kinematic constraints. Reuleaux plotted all the contact normals in the space of velocity centers, to determine whether an object is completely constrained. We use the space of *acceleration* centers, and plot the contact tangents as well as the contact normals. The lines cut the space of acceleration centers into polygons, line segments, and isolated points, each of which potentially corresponds to some contact mode. See (Brost and Mason 1989) for details and examples.

Having enumerated the contact modes, we then determine applied forces for each mode. Let f be the total force acting on the object, and let t and e be the contact force, and the effector force, respectively, so that $f = t + e$. Then given a desired total force f and a contact force t, we could calculate the effector force $e = f - t$. Now, for a given contact mode i, let F_i be the set of possible total forces f, let T_i be the set of possible contact forces t. Then we can compute the corresponding effector forces $E_i = \{f - t \mid f \in F_i, t \in T_i\}$. The complete procedure is summarized below:

1. Enumerate the contact modes $\{i\}$.

2. For each contact mode i:

   (a) Construct total forces F_i: for each object acceleration a (and angular acceleration α) consistent with contact mode i, Newton's law gives a total force $f = ma$ (and torque $\tau = I\alpha$). The set of all such forces forms a set F_i which can be described by the moment-labelling (F_{i+}, F_{i-}).

   (b) Construct task contact forces T_i, by applying Coulomb's law. Coulomb's law gives a particular

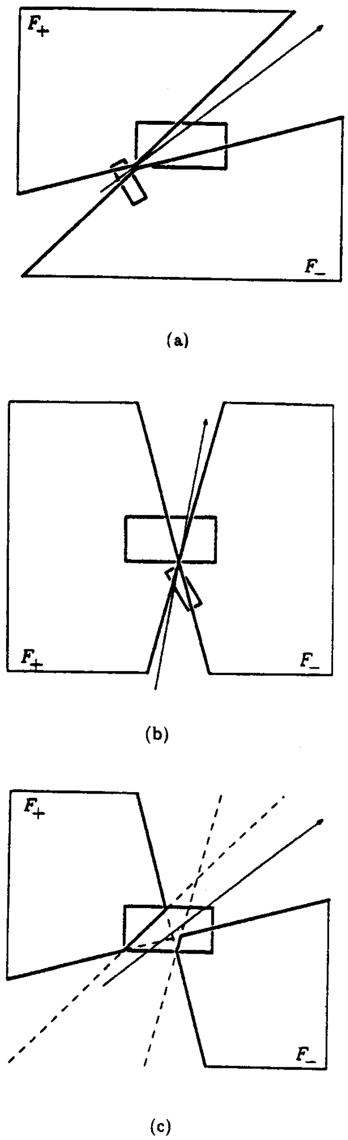

Figure 2: The resultant forces due to two frictional contacts. Each contact gives rise to a friction cone. The resultants, i.e. the positive linear combinations of the two cones, are given by intersecting the positive regions to obtain F_+, and intersecting the negative regions to obtain F_-.

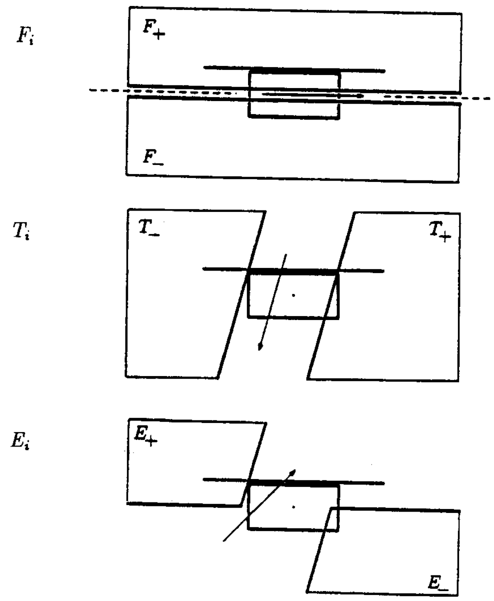

Figure 3: The block-along-wall problem. The goal is to slide the block along the wall to the right. First we construct the set F_i of total forces, described by the regions F_{i+} and F_{i-} above and below the line, respectively. Next we construct the contact forces T_i. Applying Coulomb's law, each contact gives a line of force, dividing the plane into $+$ and $-$ half-planes. Intersecting the $+$ half-planes yields T_{i+} to the right of both lines, and intersecting the $-$ half-planes yields T_{i-} to the left of both lines. Finally, we construct the effector forces $E_i = F_i \ominus T_i$, obtained by taking $E_{i+} = F_{i+} \cap T_{i-}$, and $E_{i-} = F_{i-} \cap T_{i+}$. Any effector force passing between these two regions will slide the block along the wall.

line of force for each moving contact, and a friction cone at each stationary contact. We can describe each frictional contact using the moment-labelling representation, and intersect the labeled regions to obtain (T_{i+}, T_{i-}).

(c) Construct effector forces E_i: the set of all forces e such that $e + t = f$, for some $t \in T_i$ and some $f \in F_i$. This set can also be described as the convex hull of the set F_i and the set $-T_i$, which is computed by changing the signs on the T_i moment-labelling, and intersecting: $(E_{i+}, E_{i-}) = (F_{i+} \cap T_{i-}, F_{i-} \cap T_{i+})$.

The result is a mapping from each contact mode i to a set of effector forces E_i. Figure 3 illustrates the approach by applying it to analyze the block-along-wall problem. Figure 4 shows analysis of two-point insertion of a peg in a hole.

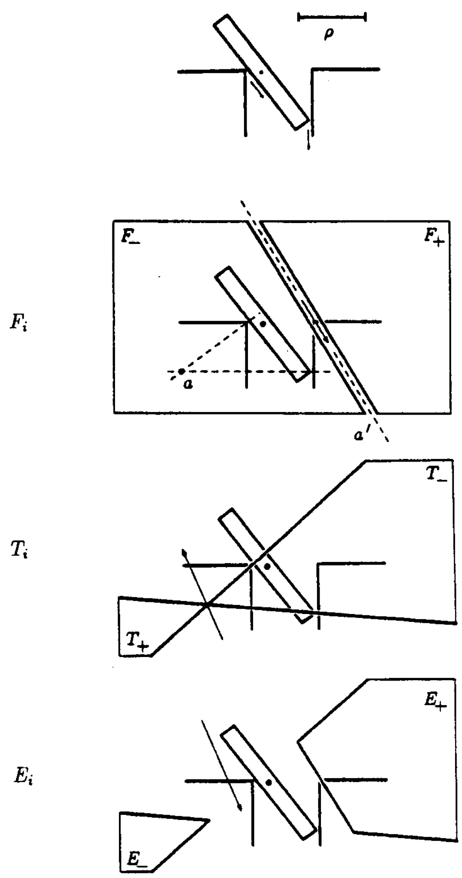

Figure 4: Two-point peg insertion. The goal is to slide the peg into the hole while maintaining two points of contact. ρ is the radius of gyration. First we construct the total forces F_i, represented by the regions F_{i+}, to the right of the line a', and F_{i-}, to the left. The point a is the desired acceleration center, and the line a' is the line of force that would cause an angular acceleration of the peg about the point a. Next we compute the contact forces T_i. Coulomb's law dictates a line of force at each contact. T_i is represented by intersecting the + and − half-planes for the two contacts to obtain the regions T_{i+} and T_{i-} shown. Finally we compute effector forces $E_i = F_i \ominus T_i$, obtained by taking $E_{i+} = F_{i+} \cap T_{i-}$, and $E_{i-} = F_{i-} \cap T_{i+}$. Any effector force passing between these two regions will move the peg down in two point contact.

3 Synthesis of robot motions

This section explores a few problems in robot motion planning, describing application of the moment-labeling method, with two examples. We consider the task domain of planar positioning and assembly problems, with Coulomb friction. The approach involves three steps:

1. Goals are first reduced to constraints on the instantaneous motion of the planar rigid body;
2. The method of the previous section is applied to obtain a set of constraints on the force applied to the object;
3. Robot commands are derived to satisfy the constraints on applied force.

This section describes an approach to the third step—deriving robot commands to satisfy given constraints on applied force.

Example 1: tilting. Here we assume that the block is against a wall in a tray, which can be tilted in any direction, or, equivalently, that the direction of the gravity vector may be controlled. We can apply any desired force, but no moment, to objects in the plane. That is, the line of applied force always passes through the object center of mass. This is represented by + and − regions each consisting of a single point at the center of mass (Figure 5). To find an action that will produce the desired motion, we can intersect the set of feasible forces with the set of goal effector forces, derived in Figure 3. Intersection of two sets is accomplished by taking the convex hull of the two + regions, and of the two − regions.

Example 2: pushing. To push the block along the wall, several cases (twenty-four) must be considered, depending on which feature of the block is pushed, and on the motion of the finger relative to the block (left slip, no slip, right slip). Of these twenty-four cases, four lead to solutions: pushing the trailing edge with left slip, and pushing the outboard trailing vertex with left slip, no slip, or right slip. Here we illustrate by analyzing a single case, where the finger is pushing the trailing edge of the block, with the finger slipping left relative to the block. The set of feasible forces can be described using the moment-labeling representation Figure 6. By intersecting with the set of goal effector forces calculated earlier in Figure 3, we obtain the solution pushing motions.

The transformation from force constraint to command parameters is especially easy in these two examples. For tilting, the direction of the line of force directly determines the tilt azimuth. And for pushing, the location of the line of force directly determines the contact location. Some other operations also fit this pattern. For compliant motions, we can use a simple linear spring as a passive compliance. The force constraint directly determines the compliant motion: the compliance center and the effector position must fall on the line

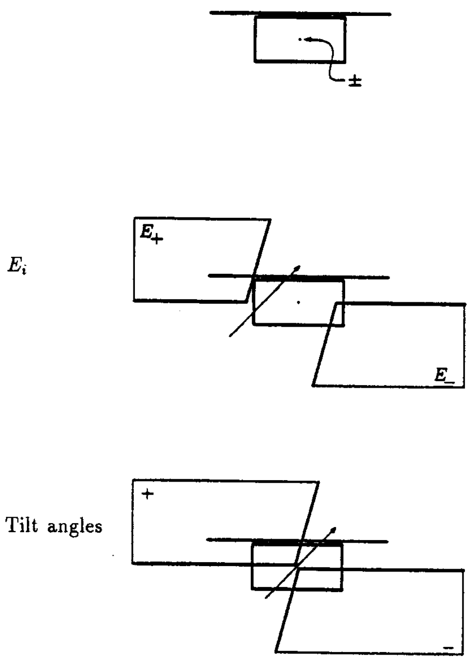

Figure 5: Tilting the block along the wall. Since gravity always acts through the center of mass, the possible "effector" forces are represented by a + region and a − region each consisting of the single point at the center of gravity. Next we show the desired effector forces E_i derived in Figure 3. To find a good tilt command, we can intersect the two force sets above, yielding the set of forces that satisfies the task-level force constraints, and can be obtained by a tilt. The intersection is obtained by taking convex hull of the + regions and of the − regions. The desired gravity vector has to pass between the resultant regions as shown. A tilt in this direction will achieve the desired motion.

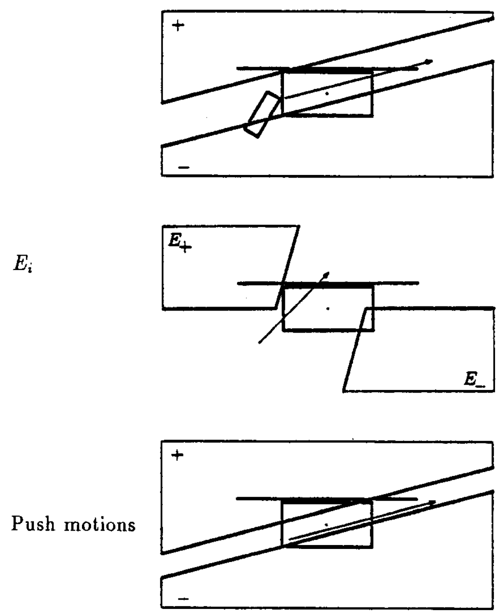

Figure 6: Pushing the block along the wall. We assume a left-sliding contact between the finger and the trailing edge of the block. The feasible pushing contacts (neglecting interference between the finger and the wall) give a set of possible pushing forces. Next we construct the desired effector forces E_i, derived in Figure 3. The solution must fall in the intersection of the two sets above, obtained by taking the convex hull of the moment-labelings. To move the block along the wall, the finger must make contact between the regions.

of force. For more general compliances, the compliance matrix might allow us to map the force constraints to effector motion constraints. For some other operations, such as finger slip, and multiple-contact pushing, the situation is not so simple, but the fundamental mechanical relationships are still most naturally expressed in terms of force. Finally, there are operations which do not seem to fit at all. Some of these are handled perfectly well by familiar programmed-motion models of action. Others, such as handling flexible objects, fit neither the conventional programmed-motion model, nor the proposed force-constraint model.

4 Conclusion

The moment-labeling representation is a particularly simple approach to problems in the mechanics of planar manipulation and planar assembly problems. The respresentation yields a multiple-contact friction cone.

The representation is suited to problems in the mechanics of manipulation, and automatic planning of manipulator programs.

The simplicity of the method, and the graphical nature, means that humans can analyze these problems by hand, and can more easily consider the implications of the analysis.

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