Definition, Determination and Characterization of Acceleration Sets for Spatial Manipulators

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Abstract

In this report the approach developed by the authors, for systematically studying the acceleration capabilities and acceleration properties of the end-effector of a planar 2 degree-of-freedom manipulator, is extended to the general spatial manipulator with three degrees-of-freedom. A central feature of this report is the determination of the properties of the quadratic mapping between the "joint-velocity" space and the acceleration space of P which then makes it possible to obtain analytical solutions for most acceleration properties of interest. We show that a fundamental way of studying these quadratic mappings is in terms of the mapping of (input) line congruences into (output) line congruences.

1 Introduction

In this paper, we apply the approach developed in (Desa and Kim, 1989-1) to the problem of determining the acceleration capability and acceleration properties of (a reference point on) the end-effector of a spatial three degree-of-freedom manipulator.

An informal statement of the problem is as follows:

Consider the general three degree-of-freedom revolute-joint manipulator shown schematically in Figure 1. We are interested in studying the acceleration of a reference point P on link 3. (P is typically a point on the joint axis of the end-effector; the acceleration of P is therefore often referred to as the end-effector acceleration). The usefulness of studying the acceleration of the end-effector has been discussed in (Yoshikawa, 1985; Khatib and Burdick, 1987; Graettinger and Krogh, 1988; Desa and Kim, 1989-2; Kim, 1989).

As shown, for example, in (Desa and Kim, 1989-1), the acceleration capability of the point P under various conditions is best described by certain acceleration sets. Two properties which are used, in general, to characterize these sets are the maximum possible magnitude of the acceleration of P and the maximum magnitude of the acceleration of P which is available in all directions. The former property is simply called the maximum acceleration of P and the latter the isotropic acceleration of P (Khatib and Burdick, 1987).

Acceleration properties of the end-effector have also been studied by (Yoshikawa, 1985; Khatib and Burdick, 1987; Graettinger and Krogh, 1988). The approach of each of these researchers has been discussed and compared with our approach in the paper (Desa and Kim, 1989-1) and we will not repeat that discussion here. We will however repeat the fundamental hypothesis underlying our approach which is as follows. By decomposing the functional relationships between the inputs (actuator torques and joint variable rates) and the output (acceleration of P) into two fundamental mappings, a linear mapping between actuator torque space and the acceleration space of point P and a quadratic (nonlinear) mapping between the "joint velocity" space and the acceleration space of P, and by deriving the properties of these two mappings, it is possible to determine the properties of all acceleration sets which are the images of the appropriate input sets under the two fundamental mappings.

The contributions of this paper are as follows:

1. The central contribution of this paper is the determination of the properties of the quadratic mapping between the joint velocity space and the acceleration space of P which then makes it possible to obtain analytical solutions for the isotropic acceleration. We show that a fundamental way of developing the properties of the quadratic mappings of interest is in terms of the mapping of (input) line congruences into (output) line congruences.

2. Closed-form analytic expressions are obtained relating important acceleration properties of manipulators to all the manipulator parameters and input variables (torques, joint variable rates or "joint velocities") of interest. (The only exception is the maximum local acceleration which is specified in terms of tight lower and upper bounds in section 6.)

3. Necessary and sufficient conditions for the existence of isotropic acceleration have been determined. (Earlier studies seem to implicitly assume that isotropic acceleration always exists.) These conditions are stated explicitly in terms of manipulator parameters and input variables.

4. Analytical expressions are derived for determining the maximum and isotropic acceleration of the end-effector at any ("local") configuration of the manipulator.

We will demonstrate the application of the theory to a particular three degree-of-freedom spatial manipulator. The application of acceleration theory to problems in manipulator design has been dealt with in (Desa and Kim, 1989-2). The next section, which describes our approach, also provides the dual function of being a "road-map" of the paper.

2 Description of the approach

The approach for studying the acceleration of (a reference point P on) the end-effector, given in (Desa and Kim, 1989) is as follows:

1. Define the input variables and output variables of interest (subsection 3.1). The output of interest is the acceleration of the reference point P.

2. Define the input sets of interest (subsection 3.1).

3. Define the input-output functional relations. These are obtained from the dynamical and kinematical equations of the manipulator (subsection 3.2).

4. Define fundamental mappings from these functional relations (subsection 3.3). There are two fundamental mappings, a linear mapping and a quadratic mapping.

5. Define the image sets of the input sets under the mappings obtained in step 4 (subsection 3.4). These image sets are the acceleration sets of interest.

6. Define general properties which can be used to characterize ("measure") acceleration sets (subsection 3.5).

7. Determine the properties of the mappings defined in step 4 (section 4).

8. Determine the acceleration sets defined in step 5 using the properties of the mappings obtained in step 7 (section 4).

9. Determine the specific properties of the acceleration sets determined in step 8 using the "measures" or general properties defined in step 6 (section 5).

10. Determine the local acceleration properties for any configuration $\mathbf{q}$ of the manipulator using the properties of the acceleration sets obtained in step 9 (section 6).

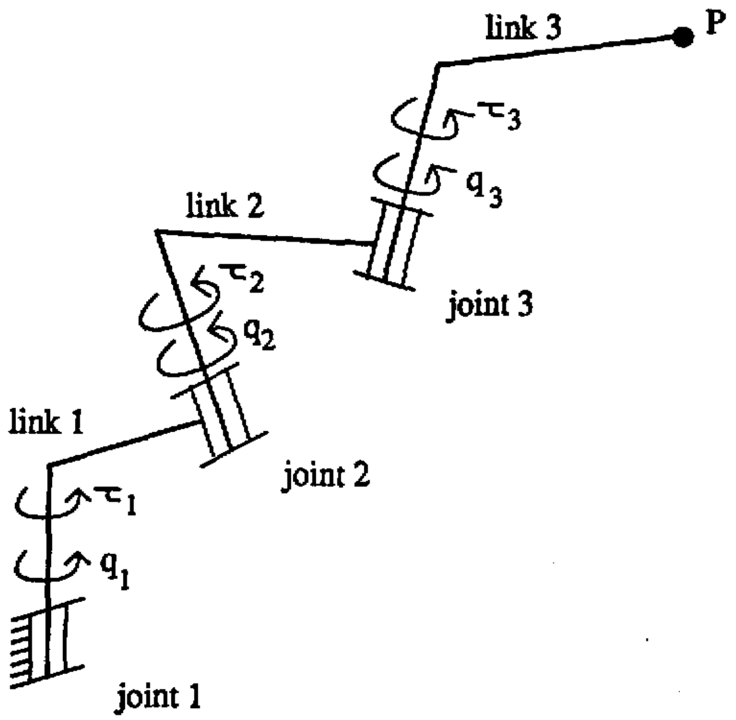

Figure 1: Schematic diagram of a general three degree-of-freedom manipulator

3 Definition of the acceleration sets

3.1 Manipulator input and output variables

Consider the general spatial three degree-of-freedom manipulator with three revolute joints shown schematically in Figure 1. In this subsection, we define the link parameters, the input variables, the input sets, the output variables and the output sets for this general spatial manipulator. The manipulator is assumed to be rigid with negligible joint friction.

The manipulator will be described by a set of geometric and inertia parameters, which will depend on the manipulator type. The geometric and inertia parameters for the spatial three degree-of-freedom manipulator of Figure 1 are also shown in Figure 9 are enumerated in the Appendix.

Next, we define the input variables, the input constraints and the corresponding input sets of the three degree-of-freedom spatial manipulator. Let q_1, q_2, and q_3 denote the generalized coordinates of the manipulator (see Figure 9), q_1, q_2 and q_3 being the joint variables, respectively, at joints 1, 2, 3. Define

$$\mathbf{q} \triangleq \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{1}$$

to be the vector of joint variables; the corresponding vector space of all $\mathbf{q}$ is called the *joint space*. If

$$q_{iL} \leq q_i \leq q_{iU}, \ i = 1,2,3 \tag{2}$$

represents the constraint on joint variable i, the workspace W of a manipulator is defined as

$$W = \{\mathbf{q} | q_{iL} \leq q_i \leq q_{iU}, \ i = 1,2,3\}. \tag{3}$$

Let $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$ denote the joint variable rates. Define

$$\dot{\mathbf{q}} \triangleq \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{4}$$

to be the vector of the joint variable rates. If

$$| \dot{q}_i | \leq \dot{q}_{io}, \ i = 1,2,3 \tag{5}$$

denotes the constraints on the joint variable rates, then we can define

$$F = \{\dot{\mathbf{q}} | \ | \dot{q}_i | \leq \dot{q}_{io}, \ i = 1,2,3\} \tag{6}$$

to be the set of all the possible joint variable rate vectors, represented by regular parallelopiped $J_1K_1L_1M_1J_2K_2L_2M_1$ in Figure 2. (We will refer to this parallelopiped as the parallelopiped F for short.)

Let τ_1, τ_2, and τ_3 denote the actuator torques, respectively, at joints 1, 2, and 3, and

$$\tau \triangleq \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \tag{7}$$

denotes the vector of actuator torque vectors. Let

$$| \tau_i | \leq \tau_{io}, \ i = 1,2,3 \tag{8}$$

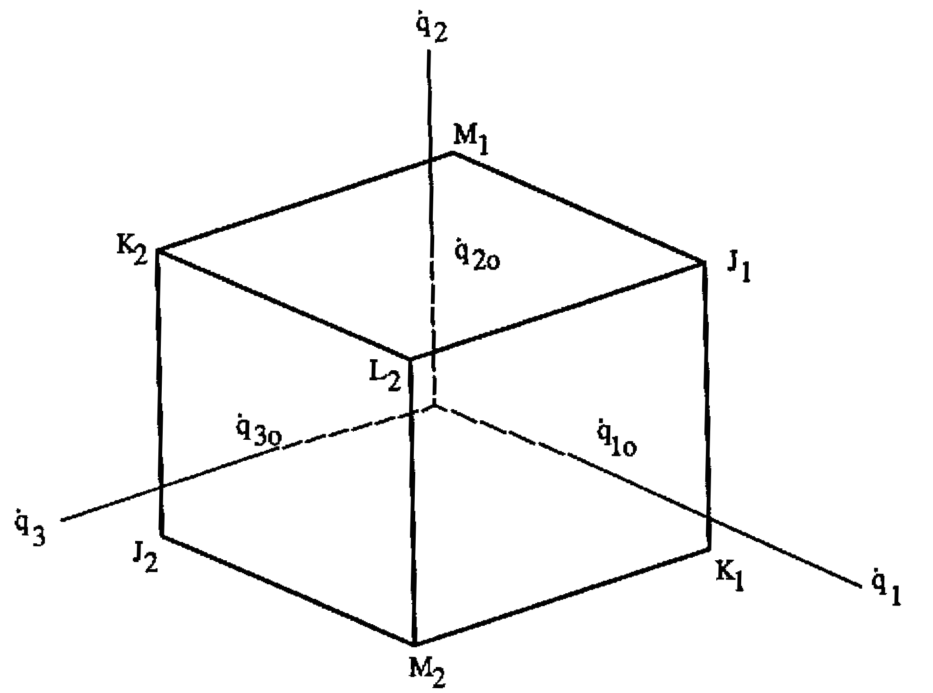

Figure 2: Set of the joint variable rates of a three degree-of-freedom manipulator

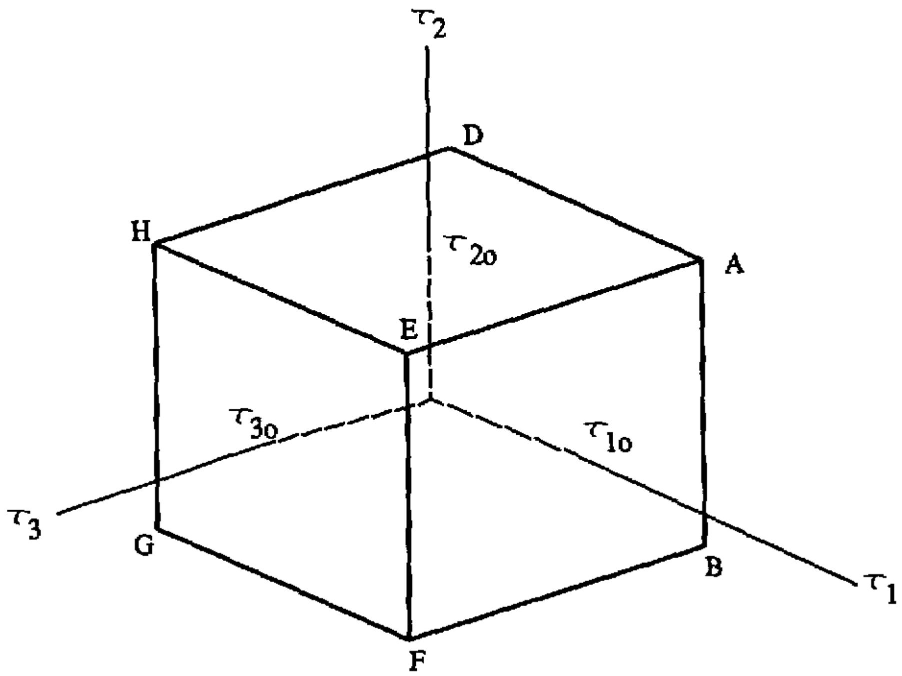

Figure 3: Set of the actuator torques of a three degree-of-freedom manipulator

denote the constraints on the actuator torques at joints 1, 2, and 3. Define

$$T = \{\tau \mid |\tau_i| \leq \tau_{io},\ i = 1,2,3\} \tag{9}$$

as the set of the allowable actuator torques, represented by regular parallelopiped *ABCDEFGH* in Figure 3. (We will refer to this parallelopiped as the parallelopiped *T* for short.)

The vectors $\mathbf{q}$, $\dot{\mathbf{q}}$ and τ will be referred to as the input variables (more precisely the input variable vectors) of the manipulator. We will also refer to the vector $\mathbf{q}$ as a configuration of the manipulator.

Let (x_1, x_2, x_3) denote the coordinates, in a reference frame fixed to the base, of a reference point P on link 3 (see Figure 1) and define

$$\mathbf{x}_p \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{10}$$

as the vector of task coordinates; the corresponding vector space of all $\ddot{\mathbf{x}}^p$ is called the task space.

The velocity $\dot{\mathbf{x}}^p$ and the acceleration $\ddot{\mathbf{x}}^p$ of the point P of the manipulator are, respectively, given by

$$\dot{\mathbf{x}}^p = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \tag{11}$$

and

$$\ddot{\mathbf{x}}^p = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}. \tag{12}$$

The acceleration of P, $\ddot{\mathbf{x}}^p$, is the output variable of interest in the present work. The corresponding vector space A of all possible $\ddot{\mathbf{x}}^p$ is called the acceleration space, expressed by

$$A = \{\ddot{\mathbf{x}} \mid \ddot{\mathbf{x}} \in R^3\}. \tag{13}$$

3.2 Functional relations between the inputs $\dot{\mathbf{q}}$, τ and the acceleration $\ddot{\mathbf{x}}^p$

The next step is to obtain the functional relations between the acceleration $\ddot{\mathbf{x}}^p$ and the inputs $\dot{\mathbf{q}}$ and τ for a given configuration $\mathbf{q}$. In this subsection, we show how the necessary functional relations can be obtained from the manipulator dynamic equations and the (so-called) manipulator Jacobian relationship.

The dynamic behavior of the most general three degree-of-freedom rigid spatial manipulator (Figure 1) can be written in the following symbolic form (Craig, 1985):

$$\mathbf{D}\ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q},\ \dot{\mathbf{q}}) + \mathbf{p} = \tau, \tag{14}$$

where $\mathbf{D}$ is the so-called mass matrix of the manipulator, $\mathbf{V}(\mathbf{q},\ \dot{\mathbf{q}})$ is the vector consisting of all terms which are non-linear in the products of the joint variable rates $\dot{q}_i$, (i = 1, 2, 3), and $\mathbf{p}$ is a vector of all terms due to gravity.

We next express non-linear terms $\mathbf{V}(\mathbf{q},\ \dot{\mathbf{q}})$ as products of a matrix and a vector. To understand how this is done, we first write $\mathbf{V}(\mathbf{q},\ \dot{\mathbf{q}})$ in its most general expanded form,

$$\mathbf{V} = \begin{bmatrix} u_{11}\dot{q}_1^2 + u_{12}\dot{q}_2^2 + u_{13}\dot{q}_3^2 + 2w_{11}\dot{q}_1\dot{q}_2 + 2w_{12}\dot{q}_2\dot{q}_3 + w_{13}\dot{q}_3\dot{q}_1 \\ u_{21}\dot{q}_1^2 + u_{22}\dot{q}_2^2 + u_{23}\dot{q}_3^2 + 2w_{21}\dot{q}_1\dot{q}_2 + 2w_{22}\dot{q}_2\dot{q}_3 + w_{23}\dot{q}_3\dot{q}_1 \\ u_{31}\dot{q}_1^2 + u_{32}\dot{q}_2^2 + u_{33}\dot{q}_3^2 + 2w_{31}\dot{q}_1\dot{q}_2 + 2w_{32}\dot{q}_2\dot{q}_3 + w_{33}\dot{q}_3\dot{q}_1 \end{bmatrix}. \tag{15}$$

Defining the two matrix operators,

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \tag{16}$$

and

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} \tag{17}$$

and two vector operators

$$(\dot{\mathbf{q}})^2 = \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \dot{q}_3^2 \end{bmatrix}, \tag{18}$$

and

$$[\dot{\mathbf{q}}]^2 = \begin{bmatrix} 2\dot{q}_1\dot{q}_2 \\ 2\dot{q}_2\dot{q}_3 \\ 2\dot{q}_1\dot{q}_1 \end{bmatrix}, \tag{19}$$

we can decompose the non-linear term $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})$ as follows:

$$\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \dot{q}_3^2 \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} \begin{bmatrix} 2\dot{q}_1\dot{q}_2 \\ 2\dot{q}_2\dot{q}_3 \\ 2\dot{q}_3\dot{q}_1 \end{bmatrix} \tag{20}$$

$$= \mathbf{U} < \dot{\mathbf{q}} >^2 + \mathbf{W}[\dot{\mathbf{q}}]^2. \tag{21}$$

Substituting equation (21) into (14), we can express the dynamic equation of a general spatial manipulator by

$$\mathbf{D}\ddot{\mathbf{q}} + \mathbf{U} < \dot{\mathbf{q}} >^2 + \mathbf{W}[\dot{\mathbf{q}}]^2 + \mathbf{p} = \tau. \tag{22}$$

This is the most general expression of describing the dynamics of a three degree-of-freedom spatial manipulator in the joint space. The matrix **D** is the mass matrix of the manipulator and the vector **p** denotes the gravitational terms which influence the dynamic behavior.

The relationship between the velocity, $\dot{\mathbf{x}}_p$, of point P, and the joint variable rate vector $\dot{\mathbf{q}}$ is well known (Desa and Roth, 1985):

$$\dot{\mathbf{x}}^p = \mathbf{J}\dot{\mathbf{q}} \tag{23}$$

where $\mathbf{J}$ is a (3×3) matrix called the manipulator Jacobian. The detailed expression of Jacobian matrix is given in the Appendix.

To obtain the expression for the acceleration $\ddot{\mathbf{x}}^p$ of the point P, we differentiate equation (23),

$$\ddot{\mathbf{x}}^p = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}. \tag{24}$$

The second term in equation (24), $\dot{\mathbf{J}}\dot{\mathbf{q}}$, can be written in the form (see the Appendix)

$$\dot{\mathbf{J}}\dot{\mathbf{q}} = -\mathbf{F} < \dot{\mathbf{q}} >^2 - \mathbf{G}[\dot{\mathbf{q}}]^2. \tag{25}$$

Substituting equations (25) into (24), we obtain

$$\ddot{\mathbf{x}}^p = \mathbf{J}\ddot{\mathbf{q}} - \mathbf{F} < \dot{\mathbf{q}} >^2 - \mathbf{G}[\dot{\mathbf{q}}]^2. \tag{26}$$

Defining the quantities,

$$\mathbf{A} = \mathbf{J}\mathbf{D}^{-1}, \tag{27}$$

$$\mathbf{B} = -\mathbf{A}\mathbf{U} - \mathbf{F}, \tag{28}$$

$$\mathbf{N} = -\mathbf{A}\mathbf{W} - \mathbf{G}, \tag{29}$$

(30)

and

$$\mathbf{s} = -\mathbf{A}\mathbf{p}, \tag{31}$$

we can easily show that the acceleration $\ddot{\mathbf{x}}^p$ of point P, obtained by combining equation (22) with equations (26) through (31), is given by

$$\ddot{\mathbf{x}}^p = \mathbf{A}\tau + \mathbf{B} < \dot{\mathbf{q}} >^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s} \tag{32}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{N}, \mathbf{s}$ are configuration dependent and have the components a_{ij}, b_{ij}, n_{ij}, s_i, (i, j = 1, 2, 3).

Equation (32) expresses the required (Input-Output) functional relation between the input variables, $\dot{\mathbf{q}}$ and τ, and the acceleration $\ddot{\mathbf{x}}^p$ of the point P (the output variable) at a given configuration $\mathbf{q}$. It is important to note that the definition of the matrix "operators" $\mathbf{U}$, $\mathbf{W}$, $\mathbf{F}$ and $\mathbf{G}$ and the vectors $< \dot{\mathbf{q}} >^2$ and $[\dot{\mathbf{q}}]^2$ enables us to write the dynamic equations in the compact form (32) which is critical in the sequel.

3.3 Mappings

In this subsection, we define two fundamental mappings between the input variables and the acceleration $\ddot{\mathbf{x}}^p$ of the point P (the output variable).

It is convenient to regard the functional relation (32) as a mapping between the input variables $\dot{\mathbf{q}}$ and τ and the output variable $\ddot{\mathbf{x}}^p$ for a given configuration $\mathbf{q}$ of the manipulator. Furthermore, defining

$$\ddot{\mathbf{x}}^p_\tau \triangleq \begin{bmatrix} \alpha_{1\tau} \\ \alpha_{2\tau} \\ \alpha_{3\tau} \end{bmatrix} = \mathbf{A}\tau \tag{33}$$

and

$$\ddot{\mathbf{x}}^p_{\dot{\mathbf{q}}} \triangleq \begin{bmatrix} \alpha_{1\dot{\mathbf{q}}} \\ \alpha_{2\dot{\mathbf{q}}} \\ \alpha_{3\dot{\mathbf{q}}} \end{bmatrix} = \mathbf{B} < \dot{\mathbf{q}} >^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}, \tag{34}$$

equation (32) can be written as

$$\ddot{\mathbf{x}}^p = \ddot{\mathbf{x}}^p_\tau + \ddot{\mathbf{x}}^p_{\dot{\mathbf{q}}}. \tag{35}$$

It is convenient to think of the vector $\ddot{\mathbf{x}}^p_\tau$ as the contribution of the torques to the acceleration of the reference point P, and the vector $\ddot{\mathbf{x}}^p_{\dot{\mathbf{q}}}$ as the contribution of the joint variable rates and gravity to the acceleration of P. Equation (35) expresses the fact that the sum of these two vectors gives us the acceleration of P for a three degree-of-freedom manipulator.

Equation (33) can be viewed as a linear, configuration-dependent, mapping between the torque vector τ and its contribution $\ddot{\mathbf{x}}^p_\tau$ to the acceleration of P. Similarly, equation (34) can be viewed as a quadratic, configuration-dependent, mapping between the joint velocity vector $\dot{\mathbf{q}}$ and its contribution $\ddot{\mathbf{x}}^p_{\dot{\mathbf{q}}}$ to the acceleration of P for a given configuration $\mathbf{q}$. These are the two mappings of interest in this section.

3.4 Manipulator acceleration sets

Having defined two fundamental mappings of interest, we are interested in the image sets of the input sets under the mappings (33) and (34) at a given configuration $\mathbf{q}$ of the manipulator. There are three image sets of interest.

3.4.1 Image set S_τ of the actuator torque set T under the linear mapping

For a given set T of the actuator torques τ described by equation (9) and represented graphically by a regular parallelopiped in the τ - space (see Figure 3), we define the image set S_τ of T under the linear mapping (33) as

$$S_\tau = \{\ddot{\mathbf{x}}^p_\tau | \ddot{\mathbf{x}}^p_\tau = \mathbf{A}\tau, \tau \in T\}. \tag{36}$$

(Note that S_τ lies in the acceleration space A.)

3.4.2 Image set $S_{\dot{\mathbf{q}}}$ of the joint variable rate set F under the quadratic mapping

For a given set F of the joint variable rates $\dot{\mathbf{q}}$ described by equation (6) and represented graphically by a regular parallelopiped (see Figure 2), we define the image set $S_{\dot{\mathbf{q}}}$ of F under the quadratic mapping as

$$S_{\dot{\mathbf{q}}} = \{\ddot{\mathbf{x}}^p_{\dot{\mathbf{q}}} | \ddot{\mathbf{x}}^p_{\dot{\mathbf{q}}} = \mathbf{B} < \dot{\mathbf{q}} >^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}, \dot{\mathbf{q}} \in F\}. \tag{37}$$

(Note that $S_{\dot{\mathbf{q}}}$ lies in the acceleration space A.) From equation (34) and the above definition (37), we see that the image set $S_{\dot{\mathbf{q}}}$ represents the set of all possible accelerations (the acceleration capability of the manipulator) when the actuators are turned off ($\tau = 0$) in any configuration $\mathbf{q}$.

3.4.3 State acceleration set

When a manipulator is in motion, the dynamic state of a manipulator can be specified by the joint variables, (q_1, q_2), and joint variable rates ($\dot{q}_1$, $\dot{q}_2$). The state vector $\mathbf{u}$ which characterizes the dynamic state of the manipulator is defined as follows:

$$\mathbf{u} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}. \tag{38}$$

For a specified dynamic state of a three degree-of-freedom manipulator, The second term of the acceleration $\ddot{\mathbf{x}}^P$ in equation (32) is a constant vector, which we denote by $\mathbf{k}(\mathbf{u})$ and define as follows:

$$\begin{aligned} \mathbf{k}(\mathbf{u}) &= \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\ &= \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3 \end{bmatrix} \\ &= \mathbf{B} < \dot{\mathbf{q}} >^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}. \end{aligned} \tag{39}$$

Equation (32) can then be written as follows:

$$\ddot{\mathbf{x}} = \mathbf{A}\tau + \mathbf{k}. \tag{40}$$

For a given dynamic state $\mathbf{u}$ of the manipulator, we define the state acceleration set $S_{\mathbf{u}}$ as the image set of T under the linear mapping (40):

$$S_{\mathbf{u}} = \{\ddot{\mathbf{x}}^P | \ddot{\mathbf{x}}^P = \mathbf{A}\tau + \mathbf{k}, \tau \in T\}. \tag{41}$$

$S_{\mathbf{u}}$ is therefore the set of all possible accelerations at any given dynamic state $\mathbf{u}$ of the manipulator. Since the dynamic state $\mathbf{u}$ of the manipulator essentially specifies the velocity $\dot{\mathbf{x}}^P$ of the point P in (11) in any configuration, we can also interpret the state acceleration set $S_{\mathbf{u}}$ (the set of available accelerations) as the acceleration capability of the manipulator when the manipulator is moving with the velocity $\dot{\mathbf{x}}^P$ in a given configuration $\mathbf{q}$.

3.5 Properties of the acceleration sets

The definitions of the acceleration sets in the previous subsection will be used in section 5 to determine them. Once these sets have been determined, one would like to characterize them.

Consider an acceleration set S in the acceleration space $\ddot{\mathbf{x}}$, and two spheres C_1 and C_2: C_1 is the smallest sphere centered at the origin which completely encloses the acceleration set and C_2 is the largest sphere centered at the origin which lies inside the acceleration set. The radius r_1 of the sphere C_1 is the maximum available acceleration in S. The radius r_2 of sphere 2 represents the largest (magnitude of) acceleration available in all directions.

We therefore define the following two properties of S:

- the maximum acceleration of S: $a_{max}(S) = r_1$,
- the isotropic acceleration of S: $a_{iso}(S) = r_2$.

Comments:

The isotropic and maximum acceleration are particularly attractive for characterizing set S, in contrast to the average acceleration, since they can be readily extracted from the dynamic equations in "closed-form" (or by appropriate bounds). The average acceleration, if required, can be numerically determined from the description of the acceleration sets given in the next section.

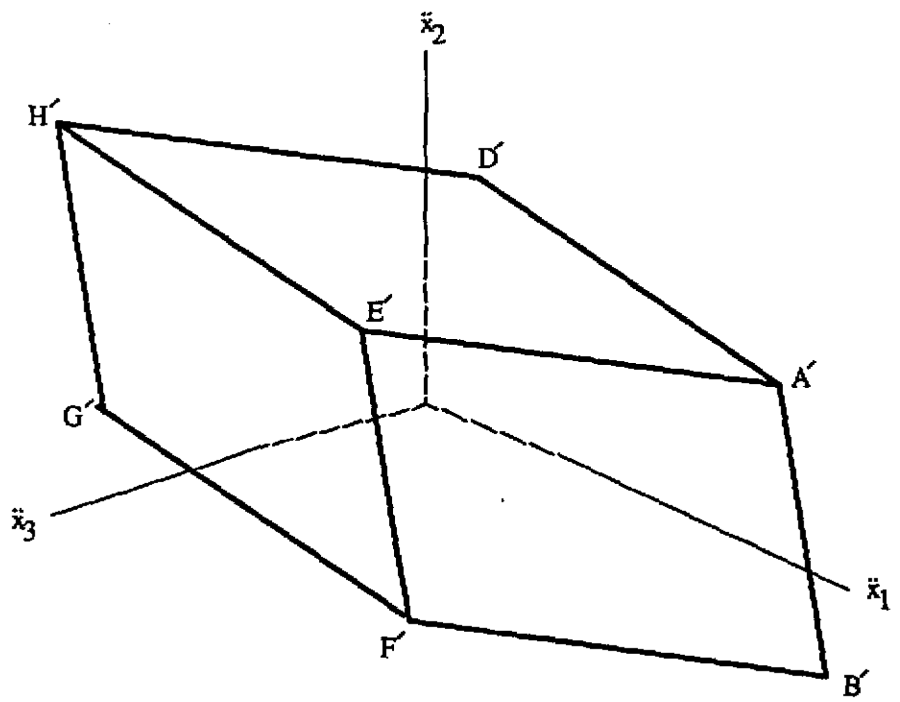

Figure 4: Image set S_τ of a three degree-of-freedom manipulator

4 Determination of the acceleration sets

Analytic expressions for the determination of the three sets S_τ, $S_{\dot{q}}$ and S_u are presented, respectively, in section 4.1, 4.2 and 4.3. The determination of S_τ and the state acceleration set S_u follows directly from well-known properties of linear mappings while the determination of the set $S_{\dot{q}}$ requires the derivation of the properties of quadratic mappings which are new The approach for determining the set $S_{\dot{q}}$ under the quadratic mapping is more fundamental than that given in (Desa and Kim, 1989).

4.1 Determination of the image set S_τ

The set S_τ is the image set of the actuator torque set T under the linear mapping (33). We determine the image set S_τ of the linear mapping of a three degree-of-freedom manipulator in the $\ddot{\mathbf{x}}$ - space. Additionally, we identify the boundaries of set S_τ, which are planes in the $\ddot{\mathbf{x}}$ - space.

Result 1: The image set S_τ of the actuator torque set T under the linear mapping (34) is (the interior and boundary of) the parallelopiped $A'B'C'D'E'F'G'H'$ in the $\ddot{x}$-space whose vertices $A', B', \ldots, H'$ are as follows:

$$A' \;:\; (a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o},\; a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o},\; a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o}) \tag{42}$$

$$B' \;:\; (a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o},\; a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o},\; a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o}) \tag{43}$$

$$C' \;:\; (-a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o},\; -a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o},\; -a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o}) \tag{44}$$

$$D' \;:\; (-a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o},\; -a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o},\; -a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o}) \tag{45}$$

$$E' \;:\; (a_{11}\tau_{1o} + a_{12}\tau_{2o} - a_{13}\tau_{3o},\; a_{21}\tau_{1o} + a_{22}\tau_{2o} - a_{23}\tau_{3o},\; a_{31}\tau_{1o} + a_{32}\tau_{2o} - a_{33}\tau_{3o}) \tag{46}$$

$$F' \;:\; (a_{11}\tau_{1o} - a_{12}\tau_{2o} - a_{13}\tau_{3o},\; a_{21}\tau_{1o} - a_{22}\tau_{2o} - a_{23}\tau_{3o},\; a_{31}\tau_{1o} - a_{32}\tau_{2o} - a_{33}\tau_{3o}) \tag{47}$$

$$G' \;:\; (-a_{11}\tau_{1o} - a_{12}\tau_{2o} - a_{13}\tau_{3o},\; -a_{21}\tau_{1o} - a_{22}\tau_{2o} - a_{23}\tau_{3o},\; -a_{31}\tau_{1o} - a_{32}\tau_{2o} - a_{33}\tau_{3o}) \tag{48}$$

$$H' \;:\; (-a_{11}\tau_{1o} + a_{12}\tau_{2o} - a_{13}\tau_{3o},\; -a_{21}\tau_{1o} + a_{22}\tau_{2o} - a_{23}\tau_{3o},\; -a_{31}\tau_{1o} + a_{32}\tau_{2o} - a_{33}\tau_{3o}) \tag{49}$$

where a_{ij}, (i,j = 1, 2, 3), are the elements of the matrix **A**. The centroid of the parallelopiped $A'B'\ldots H'$ is the origin of the $\ddot{x}$-plane (see Figure 4).

Result 2: The (planar) sides of the parallelopiped S_τ are given by the following equations:

$$A'B'F'E' \;:\; (a_{22}a_{33} - a_{23}a_{32})\ddot{x}_1 - (a_{12}a_{33} - a_{32}a_{13})\ddot{x}_2 + (a_{12}a_{23} - a_{13}a_{22})\ddot{x}_3 = \tau_{1o}\det(A) \tag{50}$$

$$D'C'G'H' \;:\; (a_{22}a_{33} - a_{23}a_{32})\ddot{x}_1 - (a_{12}a_{33} - a_{32}a_{13})\ddot{x}_2 + (a_{12}a_{23} - a_{13}a_{22})\ddot{x}_3 = -\tau_{1o}\det(A) \tag{51}$$

$$A'D'H'E' \;:\; -(a_{21}a_{33} - a_{23}a_{31})\ddot{x}_1 + (a_{11}a_{33} - a_{31}a_{13})\ddot{x}_2 - (a_{11}a_{23} - a_{13}a_{21})\ddot{x}_3 = \tau_{2o}\det(A) \tag{52}$$

$$B'C'G'F' \;:\; (a_{21}a_{33} - a_{23}a_{31})\ddot{x}_1 - (a_{11}a_{33} - a_{31}a_{13})\ddot{x}_2 + (a_{11}a_{23} - a_{13}a_{21})\ddot{x}_3 = \tau_{2o}\det(A), \tag{53}$$

$$A'B'C'D' \;:\; (a_{21}a_{32} - a_{22}a_{31})\ddot{x}_1 - (a_{11}a_{32} - a_{31}a_{12})\ddot{x}_2 + (a_{11}a_{22} - a_{12}a_{21})\ddot{x}_3 = \tau_{3o}\det(A), \tag{54}$$

$$E'F'G'H' \;:\; (a_{21}a_{32} - a_{22}a_{31})\ddot{x}_1 - (a_{11}a_{32} - a_{31}a_{12})\ddot{x}_2 + (a_{11}a_{22} - a_{12}a_{21})\ddot{x}_3 = -\tau_{3o}\det(A) \tag{55}$$

where det(A) is the determinant of the matrix **A**.

The following are well-known properties of a linear mapping:

1. A plane in the τ-space will map into a plane in the $\ddot{x}$-plane. In particular, planes p_1 ($\tau_1 = 0$), p_2 ($\tau_2 = 0$) and p_3 ($\tau_3 = 0$) map, respectively, into planes p_1', p_2' and p_3' whose equations are as follows:

$$p_1' \;:\; (a_{22}a_{33} - a_{23}a_{32})\ddot{x}_1 - (a_{12}a_{33} - a_{32}a_{13})\ddot{x}_2 + (a_{12}a_{23} - a_{13}a_{22})\ddot{x}_3 = 0, \tag{56}$$

$$p_2' \;:\; (a_{21}a_{33} - a_{23}a_{31})\ddot{x}_1 - (a_{11}a_{33} - a_{31}a_{13})\ddot{x}_2 + (a_{11}a_{23} - a_{13}a_{21})\ddot{x}_3 = 0, \tag{57}$$

$$p_3' \;:\; (a_{21}a_{32} - a_{22}a_{31})\ddot{x}_1 - (a_{11}a_{32} - a_{31}a_{12})\ddot{x}_2 + (a_{11}a_{22} - a_{12}a_{21})\ddot{x}_3 = 0. \tag{58}$$

All three planes p_1', p_2' and p_3' pass through the origin of the $\ddot{x}$-plane.

2. Any plane g_1 parallel to p_1 maps into a plane g_1' parallel to p_1'.

3. Any plane g_2 parallel to p_2 maps into a plane g_2' parallel to p_2'.

4. Any plane g_3 parallel to p_3 maps into a plane g_3' parallel to p_3'.

Proof of result 1:

By regarding the rectangular parallelopiped $AB\ldots H$ (set T) as a set of planes parallel to p_1, p_2 and p_3 one can easily show the well-known fact that the image of $AB\ldots H$ is a parallelopiped $A'B'\ldots H'$. The vertices A', B', ..., H' are the images, respectively, of the vertices A, B, ..., H which are as follows:

$$A\begin{pmatrix}\tau_{1o}\\ \tau_{2o}\\ \tau_{3o}\end{pmatrix} \quad B\begin{pmatrix}\tau_{1o}\\ -\tau_{2o}\\ \tau_{3o}\end{pmatrix} \quad C\begin{pmatrix}-\tau_{1o}\\ -\tau_{2o}\\ \tau_{3o}\end{pmatrix} \quad D\begin{pmatrix}-\tau_{1o}\\ \tau_{2o}\\ \tau_{3o}\end{pmatrix}.$$

$$E\begin{pmatrix}\tau_{1o}\\ \tau_{2o}\\ -\tau_{3o}\end{pmatrix} \quad F\begin{pmatrix}\tau_{1o}\\ -\tau_{2o}\\ -\tau_{3o}\end{pmatrix} \quad G\begin{pmatrix}-\tau_{1o}\\ -\tau_{2o}\\ -\tau_{3o}\end{pmatrix} \quad H\begin{pmatrix}-\tau_{1o}\\ \tau_{2o}\\ -\tau_{3o}\end{pmatrix}. \tag{59}$$

into equation (33), we obtain the coordinates of the vertices A', B', ..., H' as given in equation (49). From (49), we see that the vertices A' and G' are equidistant from the origin and so are the pairs (B', H'), (C', E') and (D', F'). Therefore, the origin of the $\ddot{x}$-space is the centroid of the parallelopiped $A'B'\ldots H'$.

Proof of result 2:

We next need to determine the equation of the planes $A'B'F'E'$, $D'C'G'H'$, $A'D'H'E'$, $B'C'G'F'$, $A'B'C'D'$ and $E'F'G'H'$ which form the boundary of the parallelopiped $A'B'\ldots H'$ in the $\ddot{x}$-space. The plane $A'B'F'E'$ in the $\ddot{x}$-space is the image of the plane $ABFE$ whose equation is $\tau_1 = \tau_{1o}$ in the τ-space; to obtain the equation of $A'B'F'E'$, substitute the equation of $ABFE$ ($\tau_1 = \tau_{1o}$) into (33) to obtain the following parametric equations in τ_2 and τ_3:

$$\ddot{x}_1 = a_{11}\tau_{1o} + a_{12}\tau_2 + a_{13}\tau_3 \tag{60}$$

$$\ddot{x}_2 = a_{21}\tau_{1o} + a_{22}\tau_2 + a_{23}\tau_3 \tag{61}$$

$$\ddot{x}_3 = a_{31}\tau_{1o} + a_{32}\tau_2 + a_{33}\tau_3. \tag{62}$$

Eliminating the parameter τ_2 and τ_3 between equations (60), (61) and (62), we obtain the equations of the plane $A'B'F'E'$ as given by equation (50). In a similar fashion, we obtain the equations of planes $D'C'G'H'$, $A'D'H'E'$, $B'C'G'F'$, $A'B'C'D'$, and $E'F'G'H'$ as in equations (51) through (55).

4.2 Determination of the image set $S_{\dot{q}}$

The set $S_{\dot{q}}$ is the image set of the joint rate set F under mapping (34) for a three degree-of-freedom manipulator. We decompose the set F (Figure 5 (a)) into 3 subsets F_1, F_2 and F_3 described as follows:

Definition 1: The set F_1 is the truncated line congruence (Semple and Kneebone, 1952) consisting of the doubly infinite set of line segments passing through the origin with one endpoint on the plane $J_1K_1M_2L_2$ and the other endpoint on the plane $M_1L_1J_2K_2$. A typical member of F_1 is the line segment g_1 shown in Figure 5 (b).

Definition 2: The set F_2 is the truncated line congruence consisting of the doubly infinite set of line segments passing through the origin with one endpoint on the plane $J_1L_2K_2M_1$ and the other endpoint on the plane $K_1M_2J_2L_1$. A typical member of F_2 is the line segment g_2 shown in Figure 5 (c).

Definition 3: The set F_3 is the truncated line congruence consisting of the doubly infinite set of line segments passing through the origin with one endpoint on the plane $J_1K_1L_1M_1$ and the other endpoint on the plane $L_2M_2J_2K_2$. A typical member of F_3 is the line segment g_3 shown in Figure 5 (d).

We can now state the useful results which analytically describe $S_{\dot{q}}$, the image of F.

Result 1:

1.(a) Every line of the type g_1 belonging to set F_1 maps into a line g_1' in the $\ddot{x}$-space (Figure 6 (a)), one endpoint of which is the point S whose coordinates s_i, i = 1, 2, 3 are given by (40) and the other

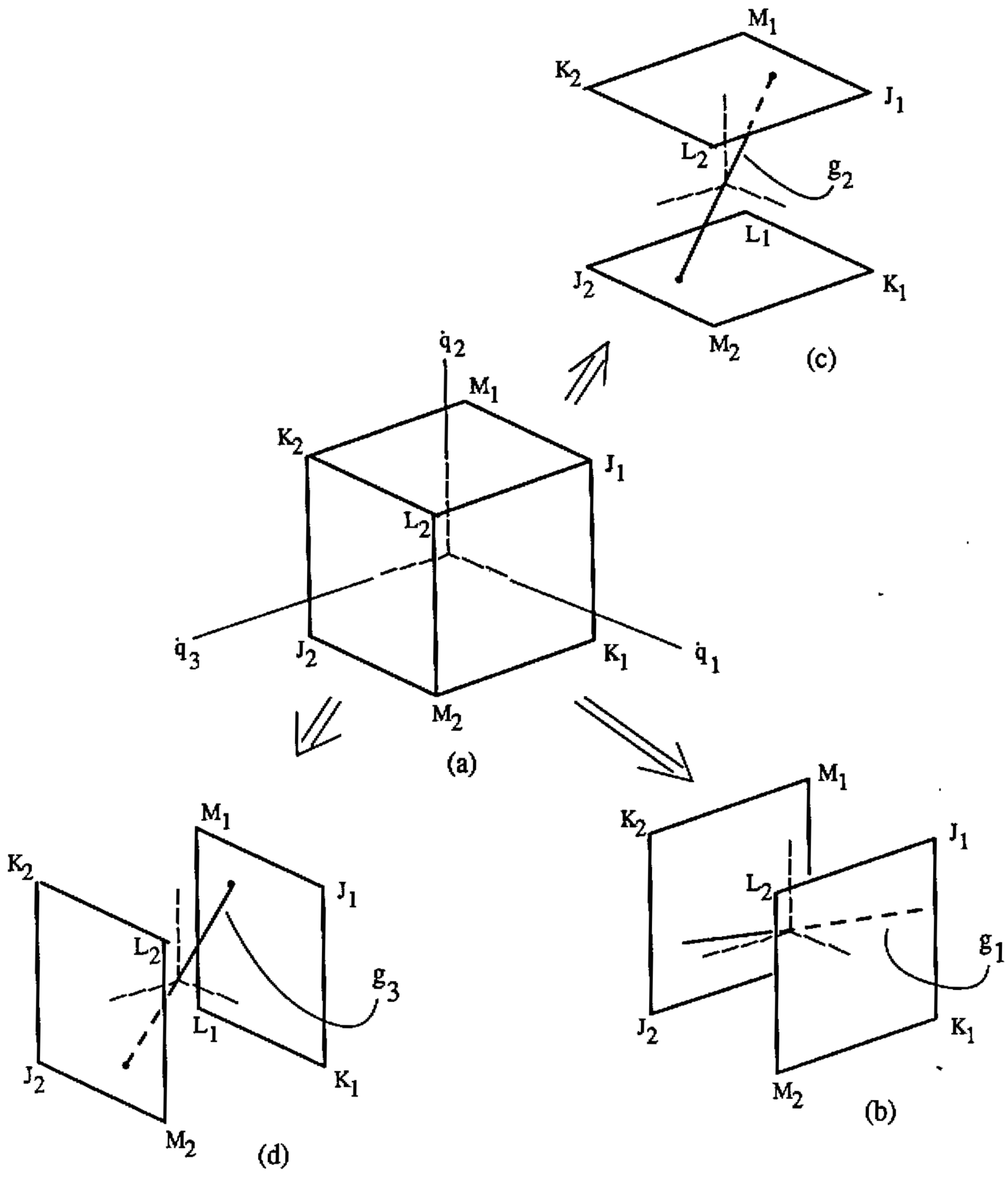

Figure 5: Image set $S_{\mathbf{q}}$ of a three degree-of-freedom manipulator

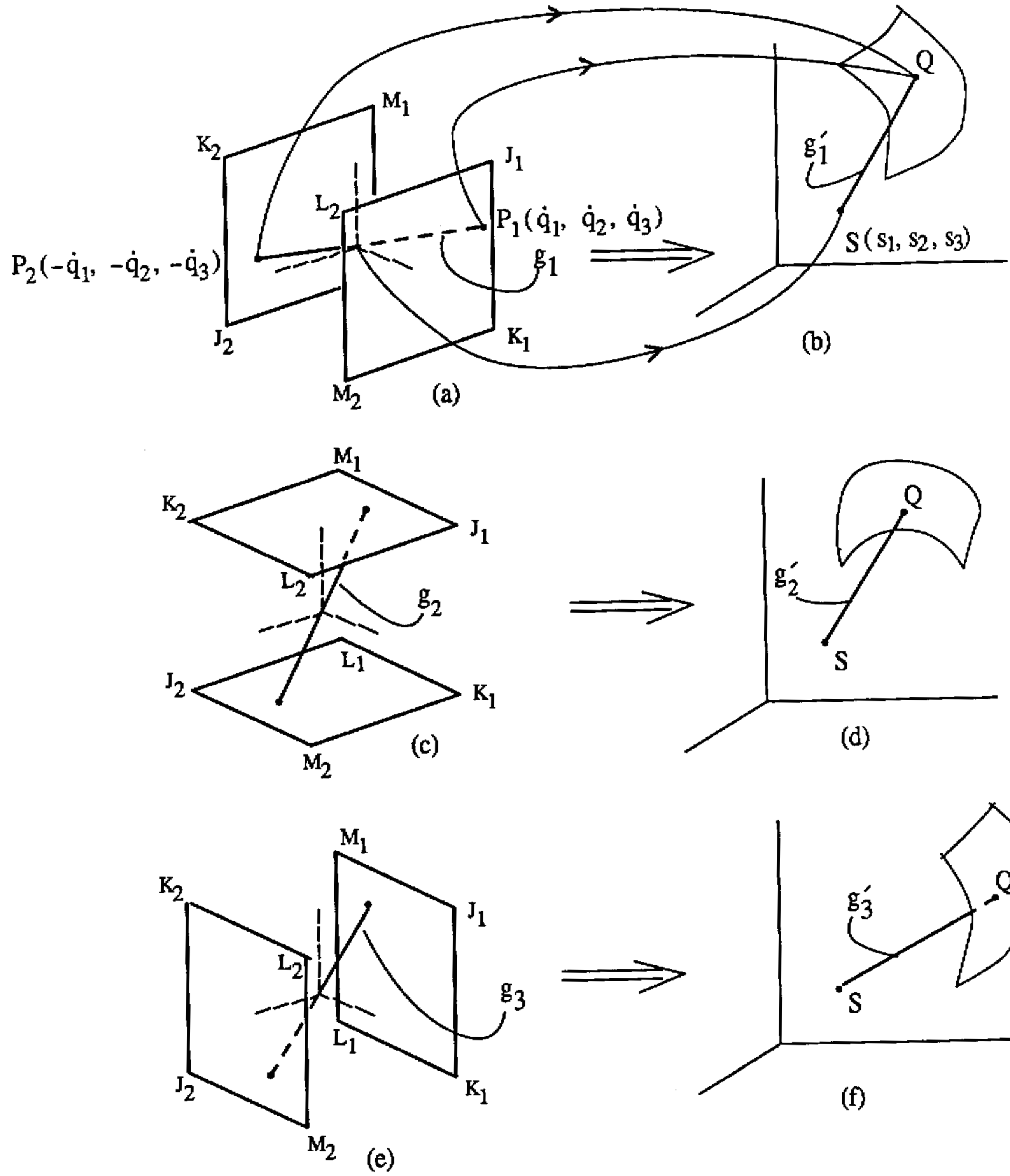

Figure 6: Quadratic mappings of a three degree-of-freedom manipulator

endpoint of which lies on the quadratic surface patch (Figure 6 (b)) whose parametric equation (in $\dot{q}_2$ and $\dot{q}_3$) is:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_{1o}^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_{1o}\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_{1o} + s_1 \\ b_{21}\dot{q}_{1o}^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_{1o}\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_{1o} + s_2 \\ b_{31}\dot{q}_{1o}^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_{1o}\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_{1o} + s_3 \end{bmatrix}. \tag{63}$$

where

$$-\dot{q}_{1o} < \dot{q}_2 < \dot{q}_{2o}$$

$$-\dot{q}_{2o} < \dot{q}_3 < \dot{q}_{3o}$$

1.(b) The set F_1 maps into a set $(S_{\dot{q}})_1$ in the $\ddot{x}$-plane which is a doubly-infinite system of line segments, one endpoint of which is the point S with coordinates s_i (i = 1, 2, 3), given by (31) and the other endpoint of which lies on the quadratic surface described by (63).

Result 2:

2.(a) Every line of the type g_2 belonging to the set F_2 maps into a line g_2' in the $\ddot{x}$-space (see Figure 6 (c)), one endpoint of which is the point S and the other endpoint of which lies on the quadratic surface patch (Figure 6 (d)) whose parametric equation (in $\dot{q}_3$ and $\dot{q}_1$) is:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_{2o}^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_{2o} + 2n_{12}\dot{q}_{2o}\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_{2o}^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_{2o} + 2n_{22}\dot{q}_{2o}\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_{2o}^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_{2o} + 2n_{32}\dot{q}_{2o}\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3 \end{bmatrix}. \tag{64}$$

where

$$|\dot{q}_1| < \dot{q}_{1o}$$

$$|\dot{q}_2| < \dot{q}_{2o}$$

2.(b) The set F_2 maps into a set $(S_{\dot{q}})_2$ in the $\ddot{x}$-plane which is a doubly-infinite system of line segments, one endpoint of which is the point S and the other endpoint of which lies on the quadratic surface described by (64).

Result 3:

3.(a) Every line of the type g_3 belonging to the set F_3 maps into a line g_3' in the $\ddot{\mathbf{x}}$-space (see Figure 6 (e)), one end of which is the point S and the other end of which lies on the quadratic surface patch (Figure 6 (f)) whose parametric equation (in $\dot{q}_1$ and $\dot{q}_2$) is:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_{3o}^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_{3o} + 2n_{13}\dot{q}_{3o}\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_{3o}^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_{3o} + 2n_{23}\dot{q}_{3o}\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_{3o}^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_{3o} + 2n_{33}\dot{q}_{3o}\dot{q}_1 + s_3 \end{bmatrix} . \quad (65)$$

where

$$-\dot{q}_{1o} < \dot{q}_1 < \dot{q}_{1o}$$

$$-\dot{q}_{2o} < \dot{q}_2 < \dot{q}_{2o}$$

3.(b) The set F_3 maps into a set $(S_{\dot{q}})_3$ in the $\ddot{\mathbf{x}}$-plane which is a doubly-infinite system of line segments, one endpoint of which is the point S and the other endpoint of which lies on the quadratic surface described by (65).

Result 4:

The image set of $S_{\dot{q}}$ of the joint variable rate set F is the union of the sets $(S_{\dot{q}})_1$, $(S_{\dot{q}})_2$, $(S_{\dot{q}})_3$ described above.

Proof of Results 1, 2, and 3:

We will first derive certain useful properties of the quadratic mapping defined by equation (34):

$$\ddot{\mathbf{x}}^p = \mathbf{B} < \dot{\mathbf{q}} >^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}.$$

The above equation can be written in the expanded form

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3 \end{bmatrix}. \tag{66}$$

Consider the (input) $\dot{\mathbf{q}}$-space. It is convenient to think of this space as being generated by the continuous doubly-infinite set of lines (also called a line congruence) passing through the origin with parametric equations

$$\begin{cases} \dot{q}_1 = t \\ \dot{q}_2 = m_1 t \quad ; \quad -\infty < m_1 < \infty,\ -\infty < m_2 < \infty. \\ \dot{q}_3 = m_2 t \end{cases} \tag{67}$$

Each value of m_1 and m_2 gives us a member of the line congruence, a typical member of which is the line l shown in Figure 7. The image l' in the $\ddot{\mathbf{x}}$-space of the line l is obtained by substituting (67) into (66) and is described by the following parametric equations,

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} m_1' t^2 + s_1 \\ m_2' t^2 + s_2 \\ m_3' t^2 + s_3 \end{bmatrix} \tag{68}$$

where

$$m_1' = b_{11} + b_{12}m_1^2 + b_{13}m_2^2 + 2n_{11}m_1 + 2n_{12}m_1m_2 + 2n_{13}m_2$$

$$m_2' = b_{21} + b_{22}m_1^2 + b_{23}m_2^2 + 2n_{21}m_1 + 2n_{22}m_1m_2 + 2n_{23}m_2$$

$$m_3' = b_{31} + b_{32}m_1^2 + b_{33}m_2^2 + 2n_{31}m_1 + 2n_{32}m_1m_2 + 2n_{33}m_2.$$

From equation (67) and (68), one can infer the following facts:

Fact 1. The image of l, *viz.* l', is a straight line.

Fact 2. The origin of the $\dot{\mathbf{q}}$-space maps into the point S of the $\ddot{\mathbf{x}}$-space.

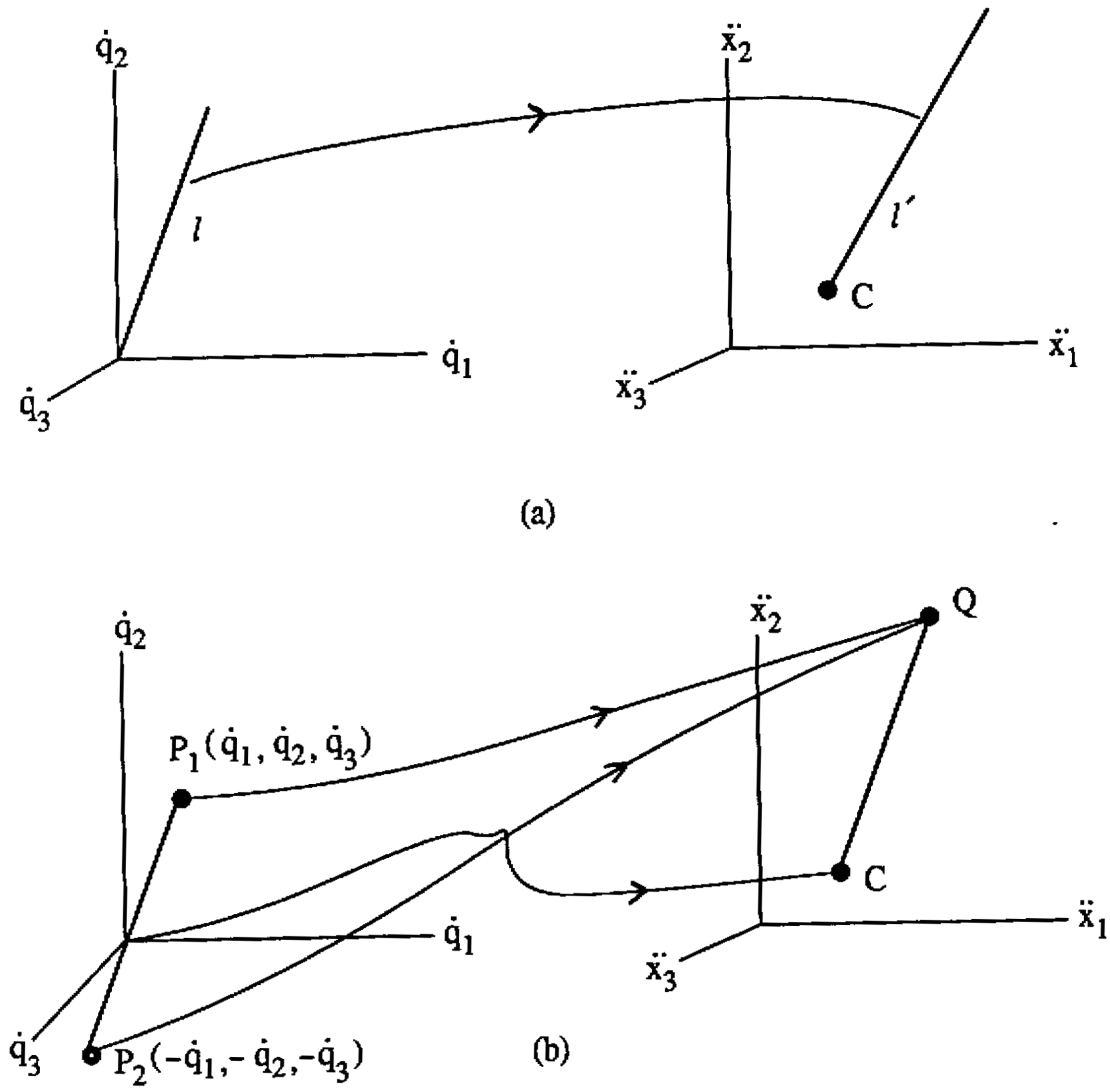

Figure 7: Properties of the quadratic mapping for a three degree-of-freedom manipulator

Fact 3. Two points with coordinates ($\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$) and ($-\dot{q}_1$, $-\dot{q}_2$, $-\dot{q}_3$) map into the same point of the $\ddot{\mathbf{x}}$-space.

These results are shown graphically in Figure 7.

Fact 1 follows from the fact that (68) is the equation of a straight line in the parameter t^2. Fact 2 follows from the fact that the point (0, 0, 0) in the $\dot{\mathbf{q}}$-space, represented by the parameter $t = 0$ in (67), maps into the point (s_1, s_2, s_3) in the $\ddot{\mathbf{x}}$-space. If t is the parameter corresponding to the point ($\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$) in the $\dot{\mathbf{q}}$-space, then, from (67), $-t$ is the parameter of the point ($-\dot{q}_1$, $-\dot{q}_2$, $-\dot{q}_3$). From (68), we see that points with parameters t and $-t$ will map into the same point in the $\ddot{\mathbf{x}}$-space. This proves Fact 3.

The following two important properties of the quadratic mapping (33) (or (66)) follow directly from the above facts:

Property 1: The image of a line l passing through the origin of the $\dot{\mathbf{q}}$-space is the half-line l', one endpoint of which is the point $S(s_1, s_2, s_3)$ of the $\ddot{\mathbf{x}}$-space (see Figure 7 (a)).

Property 2: Consider a line segment g passing through the origin of the $\dot{\mathbf{q}}$-space and with endpoints $P_1(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ and $P_2(-\dot{q}_1, -\dot{q}_2, -\dot{q}_3)$ corresponding, respectively, to parameters t and $-t$; g maps into a line segment g' in the $\ddot{\mathbf{x}}$-plane, with one endpoint at $S(s_1, s_2, s_3)$ and the other endpoint at Q whose coordinates are given by (68) (see Fig 6 (b)). Q is the image of both points P_1 and P_2.

Property 1 is basically a statement of the fundamental "folding" property of the quadratic mapping. Property 2 is more useful for our purposes.

We now determine the image, under the mapping (34), of the set F_1 which consists of the doubly-infinite system of line segment of the type g_1, (see Figure 6 (a)), which passes through the origin and which has endpoints P_1 and P_2, respectively, on planes $J_1K_1M_2L_2$ and $M_1L_1J_2K_2$ (Figure 6 (a)).

The plane $J_1K_1M_2L_2$ is described by

$$\dot{q}_1 = \dot{q}_{1o} \tag{69}$$

and the plane $M_1L_1J_2K_2$ is described by

$$\dot{q}_1 = -\dot{q}_{1o}. \tag{70}$$

Therefore, if P_1 lying on $J_1K_1M_2L_2$ has coordinates $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$, then P_2 lying on $M_1L_1J_2K_2$ has coordinates $(-\dot{q}_{1o}, -\dot{q}_2, -\dot{q}_3)$. By property 2 of the quadratic mapping, the line segment g_1 with endpoints P_1 and P_2 will map into a line segment with one endpoint at $S(s_1, s_2, s_3)$ and the other endpoint at Q (Figure 7), which is the image of both P_1 and P_2 and which we need to determine next. For every point $P_1(\dot{q}_{1o}, \dot{q}_2, \dot{q}_3)$ lying in the plane $J_1K_1M_2L_2$, there is a point $P_2(-\dot{q}_{1o}, -\dot{q}_2, -\dot{q}_3)$ lying in the plane $M_1L_1J_2K_2$ which, by Fact 3 established above, has the same image as P_1. Therefore, planes $J_1K_1M_2L_2$ and $M_1L_1J_2K_2$ have the same image. It is sufficient therefore to determine the image of plane $J_1K_1M_2L_2$. Since plane $J_1K_1M_2L_2$ is the set of all possible P_1, the image of $J_1K_1M_2L_2$ is the set of images of all possible P_1. To obtain the image of $J_1K_1M_2L_2$, we substitute its equation (69) into (66) to obtain (63) which, because it is quadratic in the parameters $\dot{q}_1$ and $\dot{q}_2$, represents a quadratic surface in the $\ddot{x}$-plane.

The quadratic surface (63) is the image of the plane $M_1L_1J_2K_2$ as well as the image of the plane $J_1K_1M_2L_2$. Any point P_1 of $M_1L_1J_2K_2$ with coordinates $(\dot{q}_{1o}, \dot{q}_2, \dot{q}_3)$ and any point P_2 of $J_1K_1M_2L_2$ with coordinates $(-\dot{q}_{1o}, -\dot{q}_2, -\dot{q}_3)$ will have the same image Q with coordinates $(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)$ given by (68).

We have thus shown that the line segment with the endpoints P_1 and P_2 will map into a line segment in the $\ddot{x}$-plane with one endpoint at $S(s_1, s_2, s_3)$ and the other endpoint Q lying on the quadratic surface (63). This completes Result 1(a).

It is now a simple matter to determine the image $(S_{\dot{q}})_1$ of F_1. By Result 1(a), the doubly-infinite set of line segments F_1 of the type g_1 with endpoints $P_1(\dot{q}_{1o}, \dot{q}_2, \dot{q}_3)$ and $P_2(-\dot{q}_{1o}, -\dot{q}_2, -\dot{q}_3)$ lying, respectively, in the planes $M_1L_1J_2K_2$ and $J_1K_1M_2L_2$ will map into the doubly-infinite set of line segments $(S_{\dot{q}})_1$ with one endpoint (always) at S and the other endpoint on the quadratic surface (63). This completes the proof of Result 1(b).

In exactly similar fashion, we can show Results 2(a) and 2(b) and Results 3(a) and 3(b).

Proof of Result 4:

Since the images of F_1, F_2 and F_3 are, respectively, $(S_{\dot{q}})_1$, $(S_{\dot{q}})_2$, and $(S_{\dot{q}})_3$, the image of $F = F_1 \cup F_2 \cup F_3$ is $S_{\dot{q}} = (S_{\dot{q}})_1 \cup (S_{\dot{q}})_2 \cup (S_{\dot{q}})_3$. $(S_{\dot{q}})_1, (S_{\dot{q}})_2$ and $(S_{\dot{q}})_3$ have been defined, respectively, in Results 1(b), 2(b),

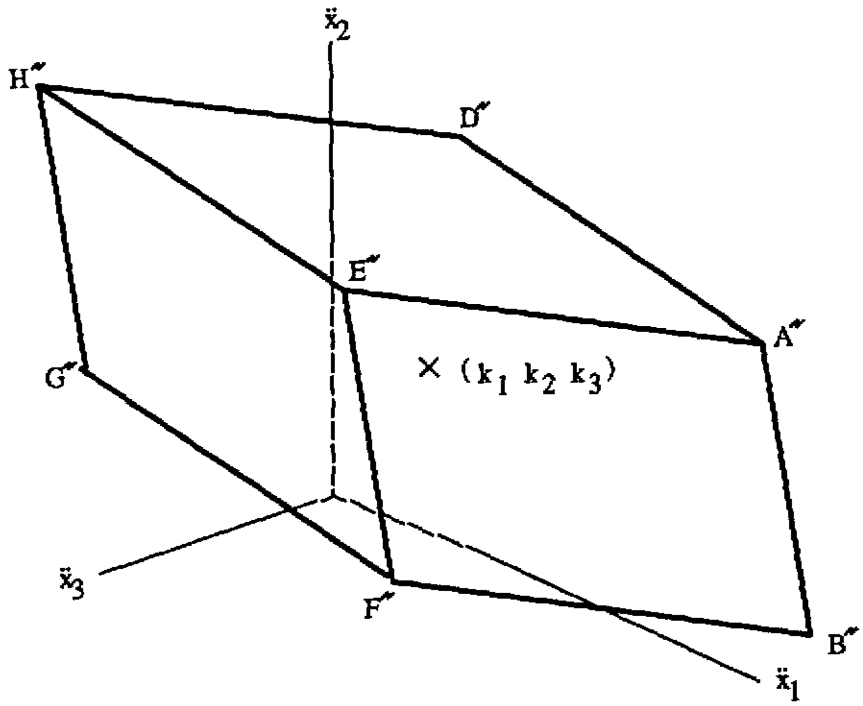

Figure 8: State acceleration set of a three degree-of-freedom manipulator

and 3(b). This completes the proof of Result 2.

Comment:

The analytical description of $(S_{\dot{q}})$ by means of $(S_{\dot{q}})_1$, $(S_{\dot{q}})_2$ and $(S_{\dot{q}})_3$ is sufficient for the extraction of the acceleration properties which we are interested in.

4.3 Determination of the state acceleration set S_u

The state acceleration S_u corresponding to a state $\mathbf{u} = (\mathbf{q}, \dot{\mathbf{q}})^T$ of the spatial manipulator was defined by equation (41) and is the image set of the actuator torque set T under the mapping (40). We obtain the following results for the state acceleration set S_u.

Result 1: For every element $\ddot{\mathbf{x}}(S_\tau)$ of the image set S_τ, there is a corresponding element $\ddot{\mathbf{x}}(S_\mathbf{u})$ of the state acceleration set $S_\mathbf{u}$, given by

$$\ddot{\mathbf{x}}(S_\mathbf{u}) = \ddot{\mathbf{x}}(S_\tau) + \mathbf{k}(\mathbf{q},\ \dot{\mathbf{q}}), \tag{71}$$

where

$$\begin{aligned} \mathbf{k}(\mathbf{q}, \dot{\mathbf{q}}) &= \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\ &= \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1 \\ b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2 \\ b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3 \end{bmatrix} \\ &= \mathbf{B} < \dot{\mathbf{q}} >^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s}. \end{aligned} \tag{72}$$

Result 2: The state acceleration set $S_\mathbf{u}$, corresponding to a state $\mathbf{u} = (\mathbf{q},\ \dot{\mathbf{q}})^T$ of the spatial three degree-of-freedom manipulator is the parallelopiped $A''B''C''D''E''F''G''H''$ shown in Figure 8 obtained by translating the set S_τ by the vector $\mathbf{k}(\mathbf{q},\ \dot{\mathbf{q}})$ in the $\ddot{\mathbf{x}}$-space. The centroid of $S_\mathbf{u}$ is (k_1, k_2, k_3).

Proof of Result 1:

The results **1** and **2** are straightforward.

From (36), a member $\ddot{\mathbf{x}}(S_\tau)$ of S_τ is given by

$$\ddot{\mathbf{x}}(S_\tau) = \mathbf{A}\tau. \tag{73}$$

From (41), a member $\ddot{\mathbf{x}}(S_\mathbf{u})$ of $S_\mathbf{u}$ is given by

$$\ddot{\mathbf{x}}(S_\mathbf{u}) = \mathbf{A}\tau + \mathbf{k} \tag{74}$$

where $\mathbf{k}$ is given by equation (72). Combining (73) and (74), we obtain

$$\ddot{\mathbf{x}}(S_\mathbf{u}) = \ddot{\mathbf{x}}(S_\tau) + \mathbf{k} \tag{75}$$

which is equation (71).

Proof of Result 2:

From equation (71), we see that if we take a vector $\ddot{\mathbf{x}}(S_T)$ of S_T and add the vector $\mathbf{k}$ to it we obtain the corresponding member $\ddot{\mathbf{x}}(S_{\mathbf{u}})$ of $S_{\mathbf{u}}$. Therefore, if we add the vector k to every vector in the set S_T we obtain the required set $S_{\mathbf{u}}$. Therefore, $S_{\mathbf{u}}$ is the parallelopiped $A''B''C''D''E''F''G''H''$ (Figure 8) obtained by translating the set S_T (the parallelopiped $A'B'C'D'E'F'G'H'$ in Figure 8) by the vector $\mathbf{k}$. The centroid of S_T is $\ddot{\mathbf{x}}(S_T) = (0, 0)$. From (75), we see that the corresponding centroid of $S_{\mathbf{u}}$ is

$$\ddot{\mathbf{x}}(S_{\mathbf{u}}) = 0 + \mathbf{k} = \mathbf{k}. \qquad (76)$$

This completes the proof of Result 2.

5 Properties of the acceleration sets

In this section, we explain how to characterize the image set, S_τ, $S_{\dot{\mathbf{q}}}$, and the state acceleration set, $S_{\mathbf{u}}$. using the results in section ??.

5.1 Properties of the acceleration set S_τ

We characterize the image set S_τ of the linear mapping as follows.

Result 1: The maximum acceleration of the acceleration set S_τ is denoted by $a_{\max}(S_\tau)$ and is given by

$$a_{\max}(S_\tau) = \max[d(OA'),\ d(OB'), d(OC'),\ d(OD')] \tag{77}$$

where

$$\begin{aligned}
d(OA') &= \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o})^2} \\
d(OB') &= \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o})^2} \\
d(OC') &= \sqrt{(-a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (-a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (-a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o})^2} \\
d(OD') &= \sqrt{(-a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (-a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (-a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o})^2}
\end{aligned}$$

Result 2: The isotropic acceleration of the acceleration set S_τ is denoted by $a_{iso}(S_\tau)$ and is given by

$$a_{iso} = [\rho(A'B'F'E'),\ \rho(A'D'H'E'),\ \rho(A'B'C'D')] \tag{78}$$

where

$$\begin{aligned}
\rho(A'B'F'E') &= \frac{|\det(\mathrm{A})|\,\tau_{1o}}{\sqrt{(a_{22}a_{33} - a_{23}a_{32})^2 + (a_{12}a_{33} - a_{13}a_{32})^2 + (a_{12}a_{23} - a_{13}a_{22})^2}}, \\
\rho(A'D'H'E') &= \frac{|\det(\mathrm{A})|\,\tau_{2o}}{\sqrt{(a_{21}a_{33} - a_{23}a_{31})^2 + (a_{11}a_{33} - a_{13}a_{31})^2 + (a_{11}a_{23} - a_{13}a_{21})^2}}, \\
\rho(A'B'C'D') &= \frac{|\det(\mathrm{A})|\,\tau_{3o}}{\sqrt{(a_{21}a_{32} - a_{22}a_{31})^2 + (a_{11}a_{32} - a_{12}a_{31})^2 + (a_{11}a_{22} - a_{12}a_{21})^2}}
\end{aligned}$$

Proof of Result 1:

The maximum acceleration of S_τ is the distance from the origin to the furthest vertex of the parallelopiped $A'B'C'D'E'F'G'H'$. Letting $d\,(O'A')$ through $d\,(O'H')$ denote, respectively, the distances of vertices A' through H' from the origin in the $\ddot{\mathbf{x}}$-space, $a_{\max}(S_\tau)$ is given by

$$a_{\max}(S_\tau) = \max[d(O'A'),\ d(O'B'),\ \ldots, d(O'H')]. \tag{79}$$

A' and G' are equidistant from the origin O'. Also, B' and H', C' and E', and D' and F' are equidistant from the origin. So, $a_{max}(S_\tau)$ is given by

$$a_{max}(S_\tau) = \max[d(O'A'), d(O'B'), d(O'C'), d(O'D')]. \tag{80}$$

Using (33) and the well-known "distance" formula, the distance $d(OA')$ from the origin O to the point A' is given by

$$d(OA') = \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o})^2}. \tag{81}$$

In exactly analogous fashion, we obtain

$$d(OB') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o})^2} \tag{82}$$

$$d(OC') = \sqrt{(-a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (-a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (-a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o})^2} \tag{83}$$

and

$$d(OD') = \sqrt{(-a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o})^2 + (-a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o})^2 + (-a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o})^2}. \tag{84}$$

Equations (80), (81), (82), (83) and (84) comprise Result 1.

Proof of Result 2:

The isotropic acceleration of S_τ is the shortest distance from the origin to the sides of the parallelopiped $A'B'C'D'E'F'G'H'$. Letting $\rho(A'B'F'E')$, $\rho(D'C'G'H')$, $\rho(A'D'H'E')$, $\rho(B'C'G'F')$, $\rho(A'B'C'D')$ and $\rho(E'F'G'H')$ denote, respectively, the distances from O' to each plane, $a_{iso}(S_\tau)$ is given by

$$a_{iso}(S_\tau) = \min[\rho(A'B'F'E'),\ \rho(D'C'G'H'),\ \rho(A'D'H'E'),\ \rho(B'C'G'F'),\ \rho(A'B'C'D'),\ \rho(E'F'G'H')]. \tag{85}$$

Since the origin is the centroid of the parallelopiped S_τ, parallel faces of the parallelopiped $A'B'C'D'E'F'G'H'$ must be equidistant from the origin. Therefore, we can write the following relations:

$$\rho(A'B'F'E') = \rho(D'C'G'H'), \tag{86}$$

$$\rho(A'D'H'E') = \rho(B'C'G'F'), \tag{87}$$

$$\rho(A'B'C'D') = \rho(E'F'G'H'). \tag{88}$$

Using (86), (87) and (88), (85) can be written as

$$a_{iso}(S_\tau) = \min[\rho(A'B'F'E'), \rho(A'D'H'E'), \rho(A'B'C'D')]. \tag{89}$$

The distance ρ from the origin to a plane $ax + by + cz + k = 0$ in the xyz - space is given by the well-known equation:

$$\rho = \frac{|k|}{\sqrt{a^2 + b^2 + c^2}}. \tag{90}$$

Using equation (90) and equations (50), (52) and (54), we obtain

$$\rho(A'B'F'E') = \frac{|\det(A)|\,\tau_{1o}}{\sqrt{(a_{22}a_{33} - a_{23}a_{32})^2 + (a_{12}a_{33} - a_{13}a_{32})^2 + (a_{12}a_{23} - a_{13}a_{22})^2}}, \tag{91}$$

$$\rho(A'D'H'E') = \frac{|\det(A)|\,\tau_{2o}}{\sqrt{(a_{21}a_{33} - a_{23}a_{31})^2 + (a_{11}a_{33} - a_{13}a_{31})^2 + (a_{11}a_{23} - a_{13}a_{21})^2}}, \tag{92}$$

$$\rho(A'B'C'D') = \frac{|\det(A)|\,\tau_{3o}}{\sqrt{(a_{21}a_{32} - a_{22}a_{31})^2 + (a_{11}a_{32} - a_{12}a_{31})^2 + (a_{11}a_{22} - a_{12}a_{21})^2}}. \tag{93}$$

Substituting (91), (92) and (93) into equation (89), we can obtain the required result (78) fot the isotropic acceleration $a_{iso}(S_\tau)$.

5.2 Properties of the acceleration set $S_{\dot{q}}$

Since each element of the set $S_{\dot{q}}$ represents the total non-linearity, we characterize the set $S_{\dot{q}}$ by the maximum magnitude element which denote the maximum non-linearity. Also, we calculate the maximum distances from direction planes in subsection 4.1 to measure the effects of the non-linearity on the state acceleration set.

Similar to a two degree-of-freedom manipulator, we illustrate the steps to the analytical expression of the furthest point of set $S_{\dot{q}}$, and the steps to the analytical expression of the furthest point from direction planes.

Definition 1: Let f_i, i = 1, 2, 3 denote, respectively, the following cubic functions in the joint variable rates $\dot{q}_i$, i = 1, 2, 3;

$$\begin{aligned} f_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) = {} & (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)(b_{11}\dot{q}_1 + n_{11}\dot{q}_2 + n_{13}\dot{q}_3) \\ & + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)(b_{21}\dot{q}_1 + n_{21}\dot{q}_2 + n_{23}\dot{q}_3) \\ & + (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3)(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) = 0, \end{aligned} \tag{94}$$

$$\begin{aligned} f_2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) = {} & (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)(b_{12}\dot{q}_1 + n_{11}\dot{q}_2 + n_{13}\dot{q}_3) \\ & + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)(b_{21}\dot{q}_1 + n_{21}\dot{q}_2 + n_{23}\dot{q}_3) \\ & + (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3)(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) = 0 \end{aligned} \tag{95}$$

Equations	Variables	Notation used to denote solutions
$f_2(\dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_3) = 0$ and $f_3(\dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_3) = 0$	$\dot{q}_2,\ \dot{q}_3$	$\dot{q}_2^{(1)},\ \dot{q}_3^{(1)}$
$f_3(\dot{q}_1,\ \dot{q}_{2o},\ \dot{q}_3) = 0$ and $f_1(\dot{q}_1,\ \dot{q}_{2o},\ \dot{q}_3) = 0$	$\dot{q}_3,\ \dot{q}_1$	$\dot{q}_3^{(2)},\ \dot{q}_1^{(2)}$
$f_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_{3o}) = 0$ and $f_2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_{3o}) = 0$	$\dot{q}_1,\ \dot{q}_2$	$\dot{q}_1^{(3)},\ \dot{q}_2^{(3)}$
$f_1(\dot{q}_1,\ \dot{q}_{2o},\ \dot{q}_{3o}) = 0$	$\dot{q}_1$	$\dot{q}_1^{(4)}$
$f_1(\dot{q}_1,\ -\dot{q}_{2o},\ \dot{q}_{3o}) = 0$	$\dot{q}_1$	$\dot{q}_1^{(5)}$
$f_2(\dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_{3o}) = 0$	$\dot{q}_2$	$\dot{q}_2^{(6)}$
$f_2(\dot{q}_{1o},\ \dot{q}_2,\ -\dot{q}_{3o}) = 0$	$\dot{q}_2$	$\dot{q}_2^{(7)}$
$f_3(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_3) = 0$	$\dot{q}_3$	$\dot{q}_3^{(8)}$
$f_3(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_3) = 0$	$\dot{q}_3$	$\dot{q}_3^{(9)}$

Table 1: Solutions of cubic equations

$$\begin{aligned} f_3 = {} & (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)(b_{11}\dot{q}_1 + n_{11}\dot{q}_2 + n_{13}\dot{q}_3) \\ & +(b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)(b_{21}\dot{q}_1 + n_{21}\dot{q}_2 + n_{23}\dot{q}_3) \\ & +(b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3)(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) = 0. \end{aligned} \tag{96}$$

where $f_i(\dot{q}_1,\ \dot{q}_1,\ \dot{q}_1)$, (i = 1, 2, 3) is cubic in $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_3$.

Definition 2: It is useful in our derivaions to be able to refer to the solutions of certain equations which play an important role in obtaining the maximum acceleration of $S_{\dot{\mathbf{q}}}$, $a_{max}(S_{\dot{\mathbf{q}}})$. Each equation or equation pair of interest is given in column 1 and the corresponding variables are indicated in column 2. All equations in column 1 are cubics in the variables in column 2. The notation used to denote the solution of each equation or equation pair is given in column 3.

Definition 3:

$$l(\dot{\mathbf{q}}) \triangleq l(\dot{q}_1,\ \dot{q}_2, \dot{q}_3)$$

$$\triangleq \begin{pmatrix} (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)^2 \\ + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)^2 \\ + (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3)^2 \end{pmatrix}^{\frac{1}{2}} . \tag{97}$$

Definition 4: Let h_i, i = 1, 2, 3 denote, respectively, the following linear equations in the joint variable rates, $\dot{q}_i$, i = 1, 2, 3;

$$h_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) = \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32})(b_{11}\dot{q}_1 + n_{11}\dot{q}_2 + n_{13}\dot{q}_3) \\ + (a_{13}a_{32} - a_{12}a_{33})(b_{21}\dot{q}_1 + n_{21}\dot{q}_2 + n_{23}\dot{q}_3) \\ + (a_{12}a_{23} - a_{13}a_{22})(b_{31}\dot{q}_1 + n_{31}\dot{q}_2 + n_{33}\dot{q}_3) \end{bmatrix} \tag{98}$$

$$h_2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) = \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32})(b_{12}\dot{q}_1 + n_{11}\dot{q}_2 + n_{12}\dot{q}_3) \\ + (a_{13}a_{32} - a_{12}a_{33})(b_{22}\dot{q}_1 + n_{21}\dot{q}_2 + n_{23}\dot{q}_3) \\ + (a_{12}a_{23} - a_{13}a_{22})(b_{32}\dot{q}_1 + n_{31}\dot{q}_2 + n_{32}\dot{q}_3) \end{bmatrix} \tag{99}$$

$$h_3(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) = \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32})(b_{13}\dot{q}_1 + n_{12}\dot{q}_2 + n_{13}\dot{q}_3) \\ + (a_{13}a_{32} - a_{12}a_{33})(b_{23}\dot{q}_1 + n_{22}\dot{q}_2 + n_{23}\dot{q}_3) \\ + (a_{12}a_{23} - a_{13}a_{22})(b_{33}\dot{q}_1 + n_{32}\dot{q}_2 + n_{33}\dot{q}_3) \end{bmatrix} . \tag{100}$$

where $h_i(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)$, (i = 1, 2, 3) is linear in $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_3$.

Definition 5: It is also useful in our derivaions to be able to refer to the solutions of certain equations which play an important role in obtaining $\rho_{\max}(\ddot{\mathrm{x}}(S_{\dot{\mathrm{q}}}), p_i)$, i = 1, 2, 3, defined below. In table 2, each equation or equation pair of interest is given in column 1 and the corresponding variables are indicated in column 2. All equations in column 1 are linear in the variables in column 2. The notation used to denote the solution of each equation or equation pair is given in column 3.

Definition 6:

$$\begin{aligned} &\sigma_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) \\ = \ &[(a_{22}a_{33} - a_{23}a_{32})^2 + (a_{12}a_{33} - a_{32}a_{13})^2 + (a_{12}a_{23} - a_{13}a_{22})^2]^{-\frac{1}{2}} \\ &\begin{pmatrix} (a_{22}a_{33} - a_{23}a_{32})2(b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1) \\ + (a_{32}a_{13} - a_{12}a_{33})2(b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2) \\ + (a_{12}a_{23} - a_{13}a_{22})2(b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3) \end{pmatrix} \\ &\sigma_2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) \end{aligned} \tag{101}$$

Equations	Variables	Notation used to denote solutions
$h_2(\dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_3) = 0$ and $h_3(\dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_3) = 0$	$\dot{q}_2,\ \dot{q}_3$	$\dot{q}_2^{[1]},\ \dot{q}_3^{[1]}$
$h_3(\dot{q}_1,\ \dot{q}_{2o},\ \dot{q}_3) = 0$ and $h_1(\dot{q}_1,\ \dot{q}_{2o},\ \dot{q}_3) = 0$	$\dot{q}_3,\ \dot{q}_1$	$\dot{q}_3^{[2]},\ \dot{q}_1^{[2]}$
$h_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_{3o}) = 0$ and $h_2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_{3o}) = 0$	$\dot{q}_1,\ \dot{q}_2$	$\dot{q}_1^{[3]},\ \dot{q}_2^{[3]}$
$h_1(\dot{q}_1,\ \dot{q}_{2o},\ \dot{q}_{3o}) = 0$	$\dot{q}_1$	$\dot{q}_1^{[4]}$
$h_1(\dot{q}_1,\ -\dot{q}_{2o},\ \dot{q}_{3o}) = 0$	$\dot{q}_1$	$\dot{q}_1^{[5]}$
$h_2(\dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_{3o}) = 0$	$\dot{q}_2$	$\dot{q}_2^{[6]}$
$h_2(\dot{q}_{1o},\ \dot{q}_2,\ -\dot{q}_{3o}) = 0$	$\dot{q}_2$	$\dot{q}_2^{[7]}$
$h_3(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_3) = 0$	$\dot{q}_3$	$\dot{q}_3^{[8]}$
$h_3(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_3) = 0$	$\dot{q}_3$	$\dot{q}_3^{[9]}$

Table 2: Solutions of linear equations

$$= [(a_{23}a_{31} - a_{21}a_{33})^2 + (a_{13}a_{31} - a_{11}a_{33})^2 + (a_{21}a_{13} - a_{11}a_{23})^2]^{-\frac{1}{2}} \begin{pmatrix} (a_{21}a_{33} - a_{23}a_{31})2(b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1) \\ +(a_{31}a_{13} - a_{11}a_{33})2(b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2) \\ +(a_{11}a_{23} - a_{13}a_{21})2(b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3) \end{pmatrix} \tag{102}$$

$$\sigma_3(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)$$

$$= (a_{32}a_{21} - a_{22}a_{31})^2 + (a_{11}a_{32} - a_{31}a_{12})^2 + (a_{11}a_{22} - a_{12}a_{21})^2]^{-\frac{1}{2}} \begin{pmatrix} (a_{32}a_{21} - a_{22}a_{31})2(b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1) \\ +(a_{31}a_{12} - a_{11}a_{32})2(b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2) \\ +(a_{11}a_{22} - a_{12}a_{21})2(b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3) \end{pmatrix} \tag{103}$$

Definition 7: Let $\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1)$, $\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2)$ and $\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3)$ denote, respectively, the distance of any point $\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}})$ of $S_{\dot{\mathbf{q}}}$ from the planes p_1, p_2 and p_3.

$$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1) \triangleq \max \rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1), \tag{104}$$

$$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2) \triangleq \max \rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_2), \tag{105}$$

$$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3) \triangleq \max \rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_3). \tag{106}$$

$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), p_1)$, for example, represents the distance of that point of $S_{\dot{\mathbf{q}}}$ furthest from plane p_1;

$\rho_{max}(\ddot{x}(S_{\dot{q}}),p_1)$, $\rho_{max}(\ddot{x}(S_{\dot{q}}),p_2)$ and $\rho_{max}(\ddot{x}(S_{\dot{q}}),p_3)$ are necessary for determining the local isotropic acceleration in subsection 5.4.

Result 1: For a general three degree-of-freedom spatial manipulator, the maximum acceleration of the acceleration set $S_{\dot{q}}$ will be denoted by $a_{max}(S_{\dot{q}})$ and is given by

$$a_{max}(S_{\dot{q}}) = \max[l_{(1)},\ l_{(2)},\ \ldots,\ l_{(13)}] \tag{107}$$

where

$$\begin{aligned}
l_{(1)} &= l(\dot{q}_{1o},\ \dot{q}_2^{(1)},\ \dot{q}_3^{(1)}) \\
l_{(2)} &= l(\dot{q}_1^{(2)},\ \dot{q}_{2o},\ \dot{q}_3^{(2)}) \\
l_{(3)} &= l(\dot{q}_1^{(3)},\ \dot{q}_2^{(3)},\ \dot{q}_{3o}) \\
l_{(4)} &= l(\dot{q}_1^{(4)},\ \dot{q}_{2o},\ \dot{q}_{3o}) \\
l_{(5)} &= l(\dot{q}_1^{(5)},\ -\dot{q}_{2o},\ \dot{q}_{3o}) \\
l_{(6)} &= l(\dot{q}_{1o},\ \dot{q}_2^{(6)},\ \dot{q}_{3o}) \\
l_{(7)} &= l(\dot{q}_{1o},\ \dot{q}_2^{(7)},\ -\dot{q}_{3o}) \\
l_{(8)} &= l(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_3^{(8)}) \\
l_{(9)} &= l(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_3^{(9)}) \\
l_{(10)} &= l(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_{3o}) \\
l_{(11)} &= l(\dot{q}_{1o},\ \dot{q}_{2o},\ -\dot{q}_{3o}) \\
l_{(12)} &= l(\dot{q}_{1o},\ -\dot{q}_{2o},\ -\dot{q}_{3o}) \\
l_{(13)} &= l(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_{3o})
\end{aligned}$$

Result 2: For a general three degree-of-freedom manipulator, the maximum distance from an element of $S_{\dot{q}}$ to the reference planes p_1, p_2 and p_3 are, respectively, given by

$$\max[\rho(\ddot{x}(S_{\dot{q}}),p_i)], i = 1,2,3 \tag{108}$$

$$= \max[(\sigma_i)_{(1)},\ (\sigma_i)_{(2)},\ \ldots,(\sigma_i)_{(13)}] \tag{109}$$

where

$$(\sigma_i)_{[1]} = \sigma_i(\dot{q}_{1o},\ \dot{q}_2^{[1]},\ \dot{q}_3^{[1]})$$
$$(\sigma_i)_{[2]} = \sigma_i(\dot{q}_1^{[2]},\ \dot{q}_{2o},\ \dot{q}_3^{[2]})$$
$$(\sigma_i)_{[3]} = \sigma_i(\dot{q}_1^{[3]},\ \dot{q}_2^{[3]},\ \dot{q}_{3o})$$
$$(\sigma_i)_{[4]} = \sigma_i(\dot{q}_1^{[4]},\ \dot{q}_{2o},\ \dot{q}_{3o})$$
$$(\sigma_i)_{[5]} = \sigma_i(\dot{q}_1^{[5]},\ -\dot{q}_{2o},\ \dot{q}_{3o})$$
$$(\sigma_i)_{[6]} = \sigma_i(\dot{q}_{1o},\ \dot{q}_2^{[6]},\ \dot{q}_{3o})$$
$$(\sigma_i)_{[7]} = \sigma_i(\dot{q}_{1o},\ \dot{q}_2^{[7]},\ -\dot{q}_{3o})$$
$$(\sigma_i)_{[8]} = \sigma_i(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_3^{[8]})$$
$$(\sigma_i)_{[9]} = \sigma_i(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_3^{[9]})$$
$$(\sigma_i)_{[10]} = \sigma_i(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_{3o})$$
$$(\sigma_i)_{[11]} = \sigma_i(\dot{q}_{1o},\ \dot{q}_{2o},\ -\dot{q}_{3o})$$
$$(\sigma_i)_{[12]} = \sigma_i(\dot{q}_{1o},\ -\dot{q}_{2o},\ -\dot{q}_{3o})$$
$$(\sigma_i)_{[13]} = \sigma_i(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_{3o})$$

where $\sigma_i(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)$ (i = 1, 2, 3) are defined by equations (101), (102) and (103).

Proof of Result 1:

The magnitude squared of the acceleration of a point $\ddot{\mathbf{x}}(S_{\dot{q}})$ of $S_{\dot{q}}$ denoted by $a^2(S_{\dot{\mathbf{q}}})$ is given by

$$\begin{aligned} a^2(S_{\dot{\mathbf{q}}}) &\triangleq l^2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) = \ddot{x}_1^2(\dot{q}_1,\ \dot{q}_2, \dot{q}_3) + \ddot{x}_2^2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) + \ddot{x}_3^2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) \\ &= (b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + b_{13}\dot{q}_3^2 + 2n_{11}\dot{q}_1\dot{q}_2 + 2n_{12}\dot{q}_2\dot{q}_3 + 2n_{13}\dot{q}_3\dot{q}_1 + s_1)^2 \\ &+ (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + b_{23}\dot{q}_3^2 + 2n_{21}\dot{q}_1\dot{q}_2 + 2n_{22}\dot{q}_2\dot{q}_3 + 2n_{23}\dot{q}_3\dot{q}_1 + s_2)^2 \\ &+ (b_{31}\dot{q}_1^2 + b_{32}\dot{q}_2^2 + b_{33}\dot{q}_3^2 + 2n_{31}\dot{q}_1\dot{q}_2 + 2n_{32}\dot{q}_2\dot{q}_3 + 2n_{33}\dot{q}_3\dot{q}_1 + s_3)^2. \end{aligned} \tag{110}$$

The maximum magnitude squared of the acceleration for the set $S_{\dot{\mathbf{q}}}$, denoted by $a^2_{\max}(S_{\dot{\mathbf{q}}})$, is given by

$$a^2_{\max}(S_{\dot{\mathbf{q}}}) = \max_{(\dot{\mathbf{q}} \in F)} l^2(\dot{q}_1,\ \dot{q}_2, \dot{q}_3), \tag{111}$$

where F is shown in Figure 2 and is specified by the constraints

$$|\dot{q}_1| \leq \dot{q}_{1o}, \tag{112}$$

$$|\dot{q}_2| \leq \dot{q}_{2o}. \tag{113}$$

$$|\dot{q}_3| \leq \dot{q}_{3o}. \tag{114}$$

The maximum of (110) will occur at $\dot{\mathbf{q}} \in F$ which is either inside F or on the boundaries of F where one, two or three constraints might be active. In section 5.1.2, we showed that "opposite" pairs of bounding planes have the same set; Using very similar arguments to those used to demonstrate the result, we can show that

1. The following pairs of bounding edges of F,

   (K_2L_2), (K_1L_1)
   (J_2M_2), (J_1M_1)
   (L_2M_2), (L_1M_1)
   (J_1K_1), (J_2K_2)
   (J_1L_2), (J_2L_1)
   (K_1M_2), (K_2M_1)

   have the same image set

2. The following pairs of vertices of F

   L_2, L_1
   J_1, J_2
   K_1, K_2
   M_2, M_1

   have the same image.

Therefore, to obtain the maximum of (110) under the constraints (112), (113) and (114), we should consider the following possibilities:

1. Neither of the constraints is active, i.e., the $\max[l^2(\dot{q}_1, \dot{q}_2, \dot{q}_3)]$ occurs at a point $\dot{\mathbf{q}}$ inside F.

2. One of the constraints (112), (113) and (114) is active, i.e., $\max[l^2(\dot{q}_1, \dot{q}_2, \dot{q}_3)]$ occurs at a point $\dot{\mathbf{q}}$ lying on the plane $J_1K_1M_2L_2$ or $J_1L_2K_2M_1$ or $L_2M_2J_2K_2$ of F.

3. Two of the constraints (112), (113) and (114) are active, i.e., $\max[l^2(\dot{q}_1, \dot{q}_2, \dot{q}_3)]$ occurs at a point $\dot{\mathbf{q}}$ lying on the edge K_2L_2, J_2M_2, L_2M_2, J_1K_1, J_1L_2 and K_1M_2 of F.

4. All of the constraints are active, i.e., $\max[l^2(\dot{q}_1, \dot{q}_2)]$ occurs at vertex L_2, vertex J_1, vertex K_1, or vertex M_2.

To obtain the conditions for each one of the above cases to yield a maximum, we first differentiate $l^2(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ with respect to $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_3$ to obtain

$$\frac{\partial l^2}{\partial \dot{q}_1} = 4f_1(\dot{q}_1, \dot{q}_2, \dot{q}_3) \tag{115}$$

$$\frac{\partial l^2}{\partial \dot{q}_2} = 4f_2(\dot{q}_1, \dot{q}_2, \dot{q}_3) \tag{116}$$

$$\frac{\partial l^2}{\partial \dot{q}_3} = 4f_3(\dot{q}_1, \dot{q}_2, \dot{q}_3) \tag{117}$$

where $f_i(\dot{q}_1, \dot{q}_2, \dot{q}_3)$, (i = 1, 2, 3), were defined in (94), (95) and (96).

Now, we consider each case.

Case 1

To obtain the maximum of l for the case where all of the constraints are inactive, we set the right-hand side of (115), (116) and (117) to zero. This gives us the equations

$$f_i(\dot{q}_1, \dot{q}_2, \dot{q}_3) = 0, \ (i = 1, 2, 3) \tag{118}$$

and the solution

$$\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0 \tag{119}$$

of which actually corresponds to the minimum value of $l^2(\dot{q}_1, \dot{q}_2, \dot{q}_3)$, *viz*, zero. Therefore, $\max(l^2)$ does not occur at a point $\dot{\mathbf{q}}$ inside F which is to be expected.

Case 2

Consider the case in which one of the constraints (112), (113) and (114) is active. When constraint (112) is active on the plane $J_1K_1M_2L_2$ of the F, we have

$$\dot{q}_1 = \dot{q}_{1o} \text{ (constant).} \tag{120}$$

To obtain the maximum of l^2, we set both $\partial l^2/\partial \dot{q}_2 = 0$ and $\partial l^2/\partial \dot{q}_3 = 0$. We therefore set the right-hand sides of both (116) and (117) to zero to obtain the following cubic equations:

$$f_2 \quad (\dot{q}_{1o}, \dot{q}_2, \dot{q}_3) = 0, \tag{121}$$

$$f_3 \quad (\dot{q}_{1o}, \dot{q}_2, \dot{q}_3) = 0. \tag{122}$$

$|\dot{q}_2| \leq \dot{q}_{2o}$, $|\dot{q}_3| \leq \dot{q}_{3o}$ whose real solution, if it exists, is denoted by $\dot{q}_2^{(1)}$ and $\dot{q}_3^{(1)}$.

Therefore, $\max l(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ for this case is given

$$\max[l(\dot{q}_1, \dot{q}_2, \dot{q}_3)] = l(\dot{q}_{1o}, \dot{q}_2^{(1)}, \dot{q}_3^{(1)}). \tag{123}$$

Comment:

Using simple arguments from algebraic geometry (Semple and Roth, 1949), we can show that if the cubics (121) and (122) with constraints $|\dot{q}_2| \leq \dot{q}_{2o}$ and $|\dot{q}_3| \leq \dot{q}_{3o}$ have real points of intersection, then they can at most one real point of intersection. If $l^2(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ does have a maximum l_{max}, then the conditions $\partial l^2/\partial \dot{q}_2 = 0$ and $\partial l^2/\partial \dot{q}_3 = 0$ for obtaining l^{max}, and therefore the pair of equations (121) and (122) which follow from them, are essentially conditions for the quadratic surface which is the image, in the $\ddot{\mathbf{x}}$-space, of the plane $J_1K_1M_2L_2$ to have a common tangent plane with a sphere of radius $l(\dot{q}_1, \dot{q}_2, \dot{q}_3)$. A sphere and a quadratic can have at most two points of tangency. Therefore, the simultaneous solutions of (121) and (122) can have at most two real roots. However, since (121) and (122) are equations of cubic curves, they will have, in general, nine points of intersection. If equations (121) and (122) had only two real roots in common, the remaining seven common roots would have to be imaginary, which is not possible. Therefore, (121) and (122) will have exactly one root, if we do not impose any constraints on $\dot{q}_2$ and $\dot{q}_3$. In the case where $\dot{q}_2$ and $\dot{q}_3$ are constrained the real root might lie outside the region specified by the constraints.

In an analogous fashion, we obtain the following maximum for l when constraint (113) is on plane $J_1L_2K_2M_1$:

$$\max[l(\dot{q}_1, \dot{q}_2, \dot{q}_3)] = l(\dot{q}_1^{(2)}, \dot{q}_{2o}, \dot{q}_3^{(2)}), \tag{124}$$

where $\dot{q}_1^{(2)}$, $\dot{q}_3^{(2)}$ is the real solution of the following two cubic equations,

$$f_1 \quad (\quad \dot{q}_1, \ \dot{q}_{2o}, \ \dot{q}_3) = 0, \tag{125}$$

$$f_3 \quad (\quad \dot{q}_1, \ \dot{q}_{2o}, \ \dot{q}_3) = 0. \tag{126}$$

We also can obtain the following maximum for l when constraint (114) is active on plane $L_2M_2J_2K_2$:

$$\max[l(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = l(\dot{q}_1^{(3)}, \ \dot{q}_2^{(3)}, \ \dot{q}_{3o}). \tag{127}$$

where $\dot{q}_1^{(3)}$, $\dot{q}_2^{(3)}$ is the real solution of the following two cubic equations,

$$f_1 \quad (\quad \dot{q}_1, \ \dot{q}_2, \ \dot{q}_{3o}) = 0, \tag{128}$$

$$f_2 \quad (\quad \dot{q}_1, \ \dot{q}_2, \ \dot{q}_{3o}) = 0. \tag{129}$$

Case 3

Consider the case in which two of the constraints (112), (113) and (114) are active. When constraints (113) and (114) are active on the edge K_2L_2 of F, we have the followig conditions,

$$\dot{q}_2 \quad = \quad \dot{q}_{2o} \text{ (constant)}, \tag{130}$$

$$\dot{q}_3 \quad = \quad \dot{q}_{3o} \text{ (constant)}. \tag{131}$$

To obtain the maximum, we set $\partial l^2/\partial \dot{q}_1 = 0$. We therefore set the right-hand side of (115) to zero and set $\dot{q}_2 = \dot{q}_{2o}$ and $\dot{q}_3 = \dot{q}_{3o}$ to obtain the cubic:

$$f_1(\dot{q}_1, \ \dot{q}_{2o}, \ \dot{q}_{3o}) = 0. \ |\dot{q}_1| \leq \dot{q}_{1o} \tag{132}$$

Using arguments similar to those used above, we can show that (132) can have at most one real solution which we denote by $\dot{q}_1^{(4)}$ The corresponding value of l is as follows:

$$\max[l(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = l(\dot{q}_1^{(4)}, \ \dot{q}_{2o}, \ \dot{q}_{3o}). \tag{133}$$

In an analogous fashion, we can obtain the following maximum for l when constraints (113) and (114) are active on edge J_2M_2:

$$\max[l(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = l(\dot{q}_1^{(5)}, \ -\dot{q}_{2o}, \ \dot{q}_{3o}). \tag{134}$$

where $\dot{q}_1^{(5)}$ is the real solution of the following cubic equation,

$$f_1 \quad (\dot{q}_1, -\dot{q}_{2o}, \dot{q}_{3o}) = 0; \tag{135}$$

For the case when constraints (112) and (114) are active on edge L_2M_2, we obtain

$$\max[l(\dot{q}_1, \dot{q}_2, \dot{q}_3)] = l(\dot{q}_{1o}, \dot{q}_2^{(6)}, \dot{q}_{3o}), \tag{136}$$

where $\dot{q}_2^{(6)}$ is the solution of the following cubic equation:

$$f_2 \quad (\dot{q}_{1o}, \dot{q}_2, \dot{q}_{3o}) = 0. \tag{137}$$

For the case when constraints (112) and (114) are active on edge J_1K_1, we obtain

$$\max[l(\dot{q}_1, \dot{q}_2, \dot{q}_3)] = l(\dot{q}_{1o}, \dot{q}_2^{(7)}, -\dot{q}_{3o}), \tag{138}$$

where $\dot{q}_2^{(7)}$ is the real solution of the following cubic equation:

$$f_2 \quad (\dot{q}_1, \dot{q}_2, -\dot{q}_{3o}) = 0. \tag{139}$$

For the case when constraints (112) and (113) are active on edge J_1L_2, we obtain

$$\max[l(\dot{q}_1, \dot{q}_2, \dot{q}_3)] = l(\dot{q}_{1o}, \dot{q}_{2o}, \dot{q}_3^{(8)}), \tag{140}$$

where $\dot{q}_3^{(8)}$ is the real solution of the following cubic equation,

$$f_3 \quad (\dot{q}_{1o}, \dot{q}_{2o}, \dot{q}_3) = 0. \tag{141}$$

For the case when constraints (112) and (113) on edge K_1M_2, we obtain

$$\max[l(\dot{q}_1, \dot{q}_2, \dot{q}_3)] = l(\dot{q}_{1o}, -\dot{q}_{2o}, \dot{q}_3^{(9)}), \tag{142}$$

where $\dot{q}_3^{(9)}$ is the real solution of the following cubic equation:

$$f_3 \quad (\dot{q}_{1o}, -\dot{q}_{2o}, \dot{q}_3) = 0. \tag{143}$$

Case 4

Consider the case in which all of the constraints (115), (116) and (117) are active. When all three constraints are active, and if $\max[l^2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)]$ occurs at $L_2(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_{3o})$, then

$$\max[l(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = l(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_{3o}). \tag{144}$$

If the maximum of l^2 occurs at $J_1(\dot{q}_{1o},\ \dot{q}_{2o},\ -\dot{q}_{3o})$, then

$$\max[l(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = l(\dot{q}_{1o},\ \dot{q}_{2o},\ -\dot{q}_{3o}). \tag{145}$$

If the maximum of l^2 occurs at $K_1(\dot{q}_{1o},\ -\dot{q}_{2o},\ -\dot{q}_{3o})$, then

$$\max[l(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = l(\dot{q}_{1o},\ -\dot{q}_{2o},\ -\dot{q}_{3o}). \tag{146}$$

If the maximum of l^2 occurs at $M_2(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_{3o})$, then

$$\max[l(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = l(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_{3o}). \tag{147}$$

Therefore, $a_{max}(S_{\dot{q}})$ ($= \max[l(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)]$) is obtained as the maximum of thirteen quantities defined by equations (123), (124) (127), (133), (134), (136), (138), (140), (142), (144), (145), (146) and (147). Thus we have demonstrated Result 1.

Proof of Result 2:

The distance of any point $\ddot{x}(S_{\dot{q}})$ of $S_{\dot{q}}$ from the line p_i, i=1, 2, 3, is given by

$$\rho(\ddot{x}(S_{\dot{q}}),p_1) \ \triangleq\ \sigma_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) \tag{148}$$

$$\rho(\ddot{x}(S_{\dot{q}}),p_2) \ \triangleq\ \sigma_2(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) \tag{149}$$

$$\rho(\ddot{x}(S_{\dot{q}}),p_3) \ \triangleq\ \sigma_3(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3). \tag{150}$$

We first wish to determine $\rho_{max}(\ddot{x}(S_{\dot{q}}),p_1)$ the distance of p_1 from that point of $S_{\dot{q}}$ furthest away from it (p_1).

$$\rho_{max}(\ddot{x}(S_{\dot{q}}),p_1) = \max_{(\dot{q}\in F)} \sigma_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) \tag{151}$$

where F is shown in Figure 2 and is specified by the constraints

$$|\dot{q}_1| \leq \dot{q}_{1o}, \tag{152}$$

$$|\dot{q}_2| \leq \dot{q}_{2o}. \tag{153}$$

$$|\dot{q}_3| \leq \dot{q}_{3o}. \tag{154}$$

The maximum of (101) which is required in (151) will occur at point $\dot{\mathbf{q}} \in F$ which is either inside F or on the boundaries of F where one or two or three constraints might be active. Using the same arguments as in Result 1 above, to obtain the maximum of (101) under the constraints (152), (153) and (154), we should consider the following possibilities:

1. Neither of the constraints is active, i.e., the $\max[\sigma_1(\dot{q}_1,\ \dot{q}_2, \dot{q}_3)]$ occurs at a point $\dot{\mathbf{q}}$ inside F.

2. One of the constraints (152), (153) and (154) is active, i.e., $\max[\sigma_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)]$ occurs at a point $\dot{\mathbf{q}}$ lying on the plane $J_1K_1M_2L_2$ or plane $J_1L_2K_2M_1$ or plane $L_2M_2J_2K_2$ of F.

3. Two of the constraints (152), (153) and (154) are active, i.e., $\max[\sigma_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)]$ occurs at a point $\dot{\mathbf{q}}$ lying on the edges K_2L_2, J_2M_2, L_2M_2, J_1K_1, J_1L_2 and K_1M_2 of F.

4. All of the constraints are active, i.e., $\max[\sigma_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)]$ occurs at a point $\dot{\mathbf{q}}$ lying on the vertex L_2, vertex J_1, vertex K_1 or vertex M_2.[1]

To obtain the conditions for each one of the above cases to yield a maximum, we first differentiate $\sigma_1(\dot{q}_1, \dot{q}_2,\ \dot{q}_3)$ with respect to $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_3$ to obtain

$$\frac{\partial \sigma_1}{\partial \dot{q}_1} = \frac{h_1}{z} \tag{155}$$

$$\frac{\partial \sigma_1}{\partial \dot{q}_2} = \frac{h_2}{z} \tag{156}$$

$$\frac{\partial \sigma_1}{\partial \dot{q}_3} = \frac{h_3}{z} \tag{157}$$

where h_i, (i = 1, 2, 3), have been defined in (98), (99) and (100) and

$$z = \sqrt{(a_{22}a_{33} - a_{23}a_{32})^2 + (a_{12}a_{33} - a_{32}a_{13})^2 + (a_{12}a_{23} - a_{13}a_{22})^2} \tag{158}$$

[1]Since, by virtue of Fact 3 of subsection 3.1.2, the vertices J_1 and J_2 have the same image, we only need to consider either J_1 or J_2: we will choose J_1. So are the vertices K_1 and K_2 and vertices M_1 and M_2.

Now, we consider each case.

Case 1

To obtain the maximum of ρ_1 for the case where all of the constraints are inactive, we set the right-hand side of (155), (156) and (157) to zero. This gives us the equations

$$h_i(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3) = 0, (i = 1,\ 2,\ 3),$$

and the solution

$$\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = 0 \tag{159}$$

of which actually corresponds to the minimum value of $\rho_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)$, *viz*, zero. Therefore, $\max(\rho_1)$ does not occur at a point $\dot{\mathbf{q}}$ inside F.

Case 2

Consider the case in which one of the constraints (152), (153) and (154) is active. When constraint (152) is active on the plane $J_1K_1M_2L_2$ of F, we have

$$\dot{q}_1 = \dot{q}_{1o} \text{ (constant).} \tag{160}$$

To obtain the maximum of ρ_1, we set both $\partial\rho_1/\partial\dot{q}_2 = 0$ and $\partial\rho_1/\partial\dot{q}_3 = 0$. We therefore set the right-hand side of both (156) and (157) to zero to obtain the following two linear equations,

$$h_2 \ (\ \dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_3) = 0, \tag{161}$$

$$h_3 \ (\ \dot{q}_{1o},\ \dot{q}_2,\ \dot{q}_3) = 0. \tag{162}$$

$$|\dot{q}_2| \le \dot{q}_{2o},\ |\dot{q}_3| \le \dot{q}_{3o}.$$

Denoting the solution $\dot{q}_2$ and $\dot{q}_3$ of (161) and (162) by $\dot{q}_2^{[1]}$, $\dot{q}_3^{[1]}$, the maximum of l for this case is given by

$$\max[\rho_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = \rho_1(\dot{q}_{1o},\ \dot{q}_2^{[1]},\ \dot{q}_3^{[1]}). \tag{163}$$

In an analogous fashion, we can obtain the following maximum for ρ_1 when constraint (153) is active on plane $J_1L_2K_2M_1$:

$$\max[\rho_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = \rho_1(\dot{q}_1^{[2]},\ \dot{q}_{2o},\ \dot{q}_3^{[2]}), \tag{164}$$

where $\dot{q}_1^{[2]}$, $\dot{q}_3^{[2]}$ is the solution of the following two linear equations,

$$h_1 \quad (\quad \dot{q}_1, \ \dot{q}_{2o}, \ \dot{q}_3) = 0, \tag{165}$$

$$h_3 \quad (\quad \dot{q}_1, \ \dot{q}_{2o}, \ \dot{q}_3) = 0. \tag{166}$$

We also can obtain the following maximum for ρ_1 when constraint (154) is active on plane $L_2M_2J_2K_2$:

$$\max[\rho_1(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = \rho_1(\dot{q}_1^{[3]}, \ \dot{q}_2^{[3]}, \ \dot{q}_{3o}). \tag{167}$$

where $\dot{q}_1^{[3]}$, $\dot{q}_2^{[3]}$ is the solution of the following two linear equations,

$$h_1 \quad (\quad \dot{q}_1, \ \dot{q}_2, \ \dot{q}_{3o}) = 0, \tag{168}$$

$$h_2 \quad (\quad \dot{q}_1, \ \dot{q}_2, \ \dot{q}_{3o}) = 0. \tag{169}$$

Case 3

Consider the case in which two of the constraints (152), (153) and (154) are active. When constraints (153) and (154) are active on the edge K_2L_2 of F, we have the following conditions,

$$\dot{q}_2 \quad = \quad \dot{q}_{2o} \text{ (constant)}, \tag{170}$$

$$\dot{q}_3 \quad = \quad \dot{q}_{3o} \text{ (constant)}. \tag{171}$$

To obtain the maximum, we set $\partial l^2/\partial \dot{q}_1 = 0$. We therefore set the right-hand side of (155) to zero and set $\dot{q}_2 = \dot{q}_{2o}$ and $\dot{q}_3 = \dot{q}_{3o}$ to obtain

$$h_1(\dot{q}_1, \ \dot{q}_{2o}, \ \dot{q}_{3o}) = 0, \ |\dot{q}_1| \leq \dot{q}_{1o}. \tag{172}$$

From equation (172), we obtain the solution which is denoted by $\dot{q}_1^{[4]}$. The corresponding value of ρ_1 is as follows:

$$\max[\rho_1(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = \rho_1(\dot{q}_1^{[4]}, \ \dot{q}_{2o}, \ \dot{q}_{3o}). \tag{173}$$

In an analogous fashion, we can obtain the following maximum for ρ_1 when constraints (153) and (154) are active on edge J_2M_2

$$\max[\rho_1(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = \rho_1(\dot{q}_1^{[5]}, \ -\dot{q}_{2o}, \ \dot{q}_{3o}). \tag{174}$$

where $\dot{q}_1^{[5]}$ is the solution of the following linear equation,

$$h_1 \quad (\quad \dot{q}_1, \ -\dot{q}_{2o}, \ \dot{q}_{3o}) = 0; \tag{175}$$

For the case when constraints (152) and (154) are active on edge L_2M_2, we obtain

$$\max[\rho_1(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = \rho_1(\dot{q}_{1o}, \ \dot{q}_2^{[6]}, \ \dot{q}_{3o}), \tag{176}$$

where $\dot{q}_2^{[6]}$ is the solution of the following linear equation,

$$h_2 \quad (\quad \dot{q}_{1o}, \ \dot{q}_2, \ \dot{q}_{3o}) = 0. \tag{177}$$

For the case when constraints (152) and (154) are active on edge J_1K_1, we obtain

$$\max[\rho_1(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = \rho_1(\dot{q}_{1o}, \ \dot{q}_2^{[7]}, \ -\dot{q}_{3o}), \tag{178}$$

where $\dot{q}_{2o}^{[7]}$ is the solution of the following linear equation:

$$h_2 \quad (\quad \dot{q}_{1o}, \ \dot{q}_2, \ -\dot{q}_{3o}) = 0. \tag{179}$$

For the case when constraints (152) and (153) are active on edge J_1L_2, we obtain

$$\max[\rho_1(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = \rho_1(\dot{q}_{1o}, \ \dot{q}_{2o}, \ \dot{q}_3^{[8]}), \tag{180}$$

where $\dot{q}_3^{[8]}$ is the solution of the following linear equation,

$$h_3 \quad (\quad \dot{q}_{1o}, \ \dot{q}_{2o}, \ \dot{q}_3) = 0. \tag{181}$$

For the case when constraints (152) and (153) on edge K_1M_2, we obtain

$$\max[\rho_1(\dot{q}_1, \ \dot{q}_2, \ \dot{q}_3)] = \rho_1(\dot{q}_{1o}, \ -\dot{q}_{2o}, \ \dot{q}_3^{[9]}), \tag{182}$$

where $\dot{q}_3^{[9]}$ is the solution of the following linear equation:

$$h_3 \quad (\quad \dot{q}_{1o}, \ -\dot{q}_{2o}, \ \dot{q}_3) = 0. \tag{183}$$

Case 4

Consider the case in which all of the constraints (152), (153) and (154) are active. When all three constraints are active, and if $\max[\rho_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)]$ occurs at $L_2(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_{3o})$, then

$$\max[\rho_1(\dot{q}_1,\ \dot{q}_2)] = \rho_1(\dot{q}_{1o},\ \dot{q}_{2o},\ \dot{q}_{3o}). \tag{184}$$

If the maximum of ρ_1 occurs at $J_1(\dot{q}_{1o},\ \dot{q}_{2o},\ -\dot{q}_{3o})$, then

$$\max[\rho_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = \rho_1(\dot{q}_{1o},\ \dot{q}_{2o},\ -\dot{q}_{3o}), \tag{185}$$

If the maximum of ρ_1 occurs at $K_1(\dot{q}_{1o},\ -\dot{q}_{2o},\ -\dot{q}_{3o})$, then

$$\max[\rho_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = \rho_1(\dot{q}_{1o},\ -\dot{q}_{2o},\ -\dot{q}_{3o}), \tag{186}$$

If the maximum of ρ_1 occurs at $M_2(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_{3o})$, then

$$\max[\rho_1(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)] = \rho_1(\dot{q}_{1o},\ -\dot{q}_{2o},\ \dot{q}_{3o}). \tag{187}$$

Therefore, $\rho_{\max}(\ddot{x}(S_{\dot{q}}),p_1)$ is obtained as the maximum of thirteen quantities defined by equations (163), (164) (167), (173), (174), (176), (178), (180), (182), (184), (185), (186) and (187). In exactly, analogous fashion, $\rho_{\max}(\ddot{x}(S_{\dot{q}}),p_2)$ and $\rho_{\max}(\ddot{x}(S_{\dot{q}}),p_3)$ are obtained as in (109). thus we have demonstrated Result 2.

5.3 Properties of the state acceleration set

Definition:

K : centroid of the acceleration set in the $\ddot{x}$-space with coordinates k_1, k_2 and k_3 given by (40).

$\rho(K,\ p_1)$: distance from point K to the reference plane p_1.

$\rho(K,\ p_2)$: distance from point K to the reference plane p_2.

$\rho(K,\ p_3)$: distance from point K to the reference plane p_3.

$\rho(A'B'F'E')$, $\rho(A''B''F''E'')$, ... : distance from the origin to plane $A'B'F'E'$, $A''B''F''E''$, ...

Result 1: The maximum acceleration corresponding to any dynamic state $\mathbf{u}$ of the manipulator is denoted by $a_{max}(S_\mathbf{u})$ and is given by

$$a_{max}(S_\mathbf{u}) = \max[d(OA''),\ d(OB''),\ d(OC''),\ d(OD''), d(OE''),\ d(OF''),\ d(OG''),\ d(OH'')] \tag{188}$$

where

$$\begin{aligned}
d(OA'') &= \sqrt{(a_{11}\tau_{1o}+a_{12}\tau_{2o}+a_{13}\tau_{3o}+k_1)^2+(a_{21}\tau_{1o}+a_{22}\tau_{2o}+a_{23}\tau_{3o}+k_2)^2+(a_{31}\tau_{1o}+a_{32}\tau_{2o}+a_{33}\tau_{3o}+k_3)^2}\\
d(OB'') &= \sqrt{(a_{11}\tau_{1o}-a_{12}\tau_{2o}+a_{13}\tau_{3o}+k_1)^2+(a_{21}\tau_{1o}-a_{22}\tau_{2o}+a_{23}\tau_{3o}+k_2)^2+(a_{31}\tau_{1o}-a_{32}\tau_{2o}+a_{33}\tau_{3o}+k_3)^2}\\
d(OC'') &= \sqrt{(a_{11}\tau_{1o}+a_{12}\tau_{2o}-a_{13}\tau_{3o}-k_1)^2+(a_{21}\tau_{1o}+a_{22}\tau_{2o}-a_{23}\tau_{3o}-k_2)^2+(a_{31}\tau_{1o}+a_{32}\tau_{2o}-a_{33}\tau_{3o}-k_3)^2}\\
d(OD'') &= \sqrt{(a_{11}\tau_{1o}-a_{12}\tau_{2o}-a_{13}\tau_{3o}-k_1)^2+(a_{21}\tau_{1o}-a_{22}\tau_{2o}-a_{23}\tau_{3o}-k_2)^2+(a_{31}\tau_{1o}-a_{32}\tau_{2o}-a_{33}\tau_{3o}-k_3)^2}\\
d(OE'') &= \sqrt{(a_{11}\tau_{1o}+a_{12}\tau_{2o}-a_{13}\tau_{3o}+k_1)^2+(a_{21}\tau_{1o}+a_{22}\tau_{2o}-a_{23}\tau_{3o}+k_2)^2+(a_{31}\tau_{1o}+a_{32}\tau_{2o}-a_{33}\tau_{3o}+k_3)^2}\\
d(OF'') &= \sqrt{(a_{11}\tau_{1o}-a_{12}\tau_{2o}-a_{13}\tau_{3o}+k_1)^2+(a_{21}\tau_{1o}-a_{22}\tau_{2o}-a_{23}\tau_{3o}+k_2)^2+(a_{31}\tau_{1o}-a_{32}\tau_{2o}-a_{33}\tau_{3o}+k_3)^2}\\
d(OG'') &= \sqrt{(a_{11}\tau_{1o}+a_{12}\tau_{2o}+a_{13}\tau_{3o}-k_1)^2+(a_{21}\tau_{1o}+a_{22}\tau_{2o}+a_{23}\tau_{3o}-k_2)^2+(a_{31}\tau_{1o}+a_{32}\tau_{2o}+a_{33}\tau_{3o}-k_3)^2}\\
d(OH'') &= \sqrt{(a_{11}\tau_{1o}-a_{12}\tau_{2o}+a_{13}\tau_{3o}-k_1)^2+(a_{21}\tau_{1o}-a_{22}\tau_{2o}+a_{23}\tau_{3o}-k_2)^2+(a_{31}\tau_{1o}-a_{32}\tau_{2o}+a_{33}\tau_{3o}-k_3)^2}
\end{aligned}$$

Result 2: The necessary and sufficient conditions for the existence of the isotropic acceleration are the following:

$$|\det(\mathrm{A})|\tau_{1o} - |(a_{22}a_{33}-a_{23}a_{32})k_1+(a_{13}a_{32}-a_{12}a_{33})k_2+(a_{12}a_{23}-a_{22}a_{13})k_3| > 0, \tag{189}$$

$$|\det(\mathrm{A})|\tau_{2o} - |(a_{21}a_{33}-a_{23}a_{31})k_1+(a_{11}a_{33}-a_{13}a_{31})k_2+(a_{11}a_{23}-a_{21}a_{13})k_3| > 0, \tag{190}$$

$$|\det(\mathrm{A})|\tau_{3o} - |(a_{21}a_{32}-a_{22}a_{31})k_1+(a_{12}a_{31}-a_{11}a_{32})k_2+(a_{11}a_{22}-a_{12}a_{21})k_3| > 0. \tag{191}$$

Result 3: The isotropic acceleration corresponding to any dynamic state $\mathbf{u}$ of the manipulator is denoted by $a_{iso}(S_\mathbf{u})$ and is given by

$$\min\begin{bmatrix}
\dfrac{|\det(A)|\tau_{1o}-|(a_{22}a_{31}-a_{23}a_{31})k_1+(a_{13}a_{32}-a_{12}a_{33})k_2+(a_{12}a_{23}-a_{22}a_{13})k_3|}{\sqrt{(a_{22}a_{31}-a_{23}a_{31})^2+(a_{13}a_{32}-a_{12}a_{33})^2+(a_{12}a_{23}-a_{22}a_{13})^2}},\\
\dfrac{|\det(A)|\tau_{2o}-|(a_{23}a_{31}-a_{21}a_{33})k_1+(a_{11}a_{33}-a_{13}a_{31})k_2+(a_{11}a_{23}-a_{21}a_{13})k_3|}{\sqrt{(a_{23}a_{31}-a_{21}a_{33})^2+(a_{11}a_{33}-a_{13}a_{31})^2+(a_{11}a_{23}-a_{21}a_{13})^2}},\\
\dfrac{|\det(A)|\tau_{3o}-|(a_{21}a_{32}-a_{22}a_{31})k_1+(a_{12}a_{31}-a_{11}a_{32})k_2+(a_{11}a_{22}-a_{12}a_{21})k_3|}{\sqrt{(a_{12}a_{32}-a_{22}a_{31})^2+(a_{12}a_{31}-a_{11}a_{32})^2+(a_{11}a_{22}-a_{12}a_{21})^2}}
\end{bmatrix} \tag{192}$$

Proof of result 1:

Let $d(OA'')$ through $d(OH'')$ denote, respectively, the distances of vertices A'' through H'' from the origin O in the $\ddot{\mathbf{x}}$-space. Then $a_{max}(S_\mathbf{u})$ is the distances of the furthest vertex of the set $S_\mathbf{u}$ which is the parallelopiped $A''B''C''D''E''F''G''H''$. Therefore, $a_{max}(S_\mathbf{u})$ is given by

$$a_{max}(S_\mathbf{u}) = \max[d(OA''),\ d(OB''),\ d(OC''),\ d(OD''),\ d(OE''),\ d(OF''),\ d(OG''),\ d(OH'')]. \tag{193}$$

Using (49), the coordinates $\ddot{x}_1(A'')$, $\ddot{x}_2(A'')$ and $\ddot{x}_3(A'')$ of vertex A'' in the $\ddot{\mathbf{x}}$-space are given by

$$\ddot{x}_1(A'') = \ddot{x}_1(A') + k_1 = a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o} + k_1, \tag{194}$$

$$\ddot{x}_2(A'') = \ddot{x}_2(A') + k_2 = a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o} + k_2, \tag{195}$$

$$\ddot{x}_3(A'') = \ddot{x}_3(A') + k_3 = a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o} + k_3. \tag{196}$$

The distance $d(OA'')$ from the origin O to the point A'' is given by

$$\sigma(O''A'') = \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o} + k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o} + k_2)^2 + (a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o} + k_3)^2}. \tag{197}$$

In exactly analogous fashion, we obtain

$$\sigma(O''B'') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o} + k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o} + k_2)^2 + (a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o} + k_3)^2} \tag{198}$$

$$\sigma(O''C'') = \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} - a_{13}\tau_{3o} - k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} - a_{23}\tau_{3o} - k_2)^2 + (a_{31}\tau_{1o} + a_{32}\tau_{2o} - a_{33}\tau_{3o} - k_3)^2} \tag{199}$$

$$\sigma(O''D'') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} - a_{13}\tau_{3o} - k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} - a_{23}\tau_{3o} - k_2)^2 + (a_{31}\tau_{1o} - a_{32}\tau_{2o} - a_{33}\tau_{3o} - k_3)^2} \tag{200}$$

$$\sigma(O''E'') = \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} - a_{13}\tau_{3o} + k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} - a_{23}\tau_{3o} + k_2)^2 + (a_{31}\tau_{1o} + a_{32}\tau_{2o} - a_{33}\tau_{3o} + k_3)^2} \tag{201}$$

$$\sigma(O''F'') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} - a_{13}\tau_{3o} + k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} - a_{23}\tau_{3o} + k_2)^2 + (a_{31}\tau_{1o} - a_{32}\tau_{2o} - a_{33}\tau_{3o} + k_3)^2} \tag{202}$$

$$\sigma(O''G'') = \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} + a_{13}\tau_{3o} - k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} + a_{23}\tau_{3o} - k_2)^2 + (a_{31}\tau_{1o} + a_{32}\tau_{2o} + a_{33}\tau_{3o} - k_3)^2} \tag{203}$$

$$\sigma(O''H'') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} + a_{13}\tau_{3o} - k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} + a_{23}\tau_{3o} - k_2)^2 + (a_{31}\tau_{1o} - a_{32}\tau_{2o} + a_{33}\tau_{3o} - k_3)^2} \tag{204}$$

Equations (193) and (197) through (204) comprises Result 1.

Proof of result 2 and 3:

The state acceleration set S_τ is the parallelopiped centered at $\mathbf{k}(\mathbf{u}) = (k_1, k_2, k_3)$, shown in Figure 8. The centroids of S_τ and $S_\mathbf{u}$ are, respectively, by O and K.

Using equations (90), (72) and (56) through (58), the distance from K to the planes p_1, p_2 and p_3 are given by

$$\rho(K,p_1) = \frac{|(a_{22}a_{33} - a_{23}a_{32})k_1 + (a_{13}a_{32} - a_{12}a_{33})k_2 + (a_{12}a_{23} - a_{22}a_{13})k_3|}{\sqrt{(a_{22}a_{31} - a_{23}a_{31})^2 + (a_{13}a_{32} - a_{12}a_{33})^2 + (a_{12}a_{23} - a_{22}a_{13})^2}} \tag{205}$$

$$\rho(K,p_2) = \frac{|(a_{21}a_{33} - a_{23}a_{31})k_1 + (a_{11}a_{33} - a_{13}a_{31})k_2 + (a_{11}a_{23} - a_{21}a_{13})k_3|}{\sqrt{(a_{23}a_{31} - a_{21}a_{33})^2 + (a_{11}a_{33} - a_{13}a_{31})^2 + (a_{11}a_{23} - a_{21}a_{13})^2}} \tag{206}$$

$$\rho(K,p_3) = \frac{|(a_{21}a_{32} - a_{22}a_{31})k_1 + (a_{12}a_{31} - a_{11}a_{32})k_2 + (a_{11}a_{22} - a_{12}a_{21})k_3|}{\sqrt{(a_{12}a_{32} - a_{22}a_{31})^2 + (a_{12}a_{31} - a_{11}a_{32})^2 + (a_{11}a_{22} - a_{12}a_{21})^2}} \tag{207}$$

The distance $\rho(K,\ p_1)$ from the centroid K of $S_\mathbf{u}$ to the plane p_1 is equal to the perpendicular distance between plane $A'B'F'E'$ and plane $A''B''F''E''$ and also between the plane $D'C'G'H'$ and plane $D''C''G''H''$.

The distance $\rho(K, p_2)$ is equal to the perpendicular distance between plane $A'D'H'E'$ and plane $A''D''H''E''$ and also between plane $B'C'G'F'$ and plane $B''C''G''F''$. The distance $\rho(K,p_3)$ is equal to the perpendicular distance between plane $E'F'G'H'$ and plane $E''F''G''H''$.

The state isotropic acceleration $a_{iso}(S_u)$ is the maximum acceleration which is available in all directions. It is therefore equal to the minimum of the distances from the origin O (of the acceleration plane) to the six planes of $A''B''C''D''E''F''G''H''$ (the set S_u).

Now, we can write the following expression for $a_{iso}(S_u)$:

$$a_{iso}(S_u) = \min[\rho(A''B''F''E''),\ \rho(A''E''H''D''),\ \rho(E''F''G''H''),\ \rho(D''C''G''H''),\ \rho(B''C''G''F''),\ \rho(A''B''C''D'')] \quad (208)$$

where $\rho(A''B''F''E'')$ is the (perpendicular) distance from O to plane $A''B''F''E''$ and similarly for $\rho(A''E''H''D'')$, $\rho(E''F''G''H'')$, $\rho(D''C''G''H'')$, $\rho(B''C''G''F'')$, $\rho(A''B''C''D'')$, all assumed positive by definition. From the geometry, we can write,

$$\rho(A''B''F''E''), \rho(D''C''G''H'') = \rho(A'B'F''E'') \pm \rho(K,\ p_1). \quad (209)$$

(Comment: For example, $\rho(A''B''F''E'') = \rho(A'B'F''E'') + \rho(K,p_1)$ and $\rho(D''C''G''H'') = \rho(D'C'G''H'') - \rho(K,p_1)$; the correct choice of signs will depend on the direction of the translation but as will be shown below we do not have to worry about the correct choice of signs.)

Similarly,

$$\rho(A''D''H''E''), \rho(B''C''G''F'') \;=\; \rho(A'D'H'E') \pm \rho(K,p_2), \quad (210)$$

$$\rho(E''F''G''H''), \rho(A''B''C''D'') \;=\; \rho(E'F'G'H') \pm \rho(K,p_3), \quad (211)$$

(The above comment holds for (210) and (211), too.)

Combining equations (208), (209), (210) and (211), we obtain

$$a_{iso}(S_u) = \min[\rho(A'B'F'E') \pm \rho(K,\ p_1),\ \rho(A'D'H'E') \pm \rho(K,\ p_2),\ \rho(E'F'G'H') \pm (K,\ p_3)]. \quad (212)$$

Since all distances $\rho()$ in the above equation are positive by definition, we can rewrite the above equation as

$$a_{iso}(S_u) = \min[\rho(A'B'F'E') - \rho(K,\ p_1),\ \rho(A'D'H'E') - \rho(K,\ p_2),\ \rho(E'F'G'H' - \rho(K,\ p_3)]. \quad (213)$$

Substituting equations (56) through (58) and (205) through (207) into (213), we obtain the required result (192).

Equation (213) clearly demonstrates that the isotropic acceleration $a_{iso}(S_{\mathbf{u}})$ for any state $\mathbf{u} \neq 0$ is less than $a_{iso}(S_\tau) = \min[\rho(A'B'F'E'), \rho(A'D'H'E'), \rho(E'F'G'H')]$. In fact, if $\rho(K, p_1)$, $\rho(K, p_2)$ and $\rho(K, p_3)$ are sufficiently large (equivalently, the "nonlinearities" k_1, k_2 and k_3 are sufficiently "large"), we may not have any isotropic acceleration. The necessary and sufficient conditions for the existence of the isotropic acceleration can be obtained either from (213) or (192). From (192), we obtain the following three necessary and sufficient conditions for the existence of the isotropic acceleration:

$$\tau_{1o}|\det(\mathrm{A})| \geq |(a_{22}a_{33} - a_{23}a_{32})k_1 + (a_{13}a_{32} - a_{12}a_{33})k_2 + (a_{12}a_{23} - a_{22}a_{13})k_3| \quad (214)$$

$$\tau_{2o}|\det(\mathrm{A})| \geq |(a_{21}a_{33} - a_{23}a_{31})k_1 + (a_{11}a_{33} - a_{13}a_{31})k_2 + (a_{11}a_{23} - a_{21}a_{13})k_3| \quad (215)$$

$$\tau_{3o}|\det(\mathrm{A})| \geq |(a_{21}a_{32} - a_{22}a_{31})k_1 + (a_{12}a_{31} - a_{11}a_{32})k_2 + (a_{11}a_{22} - a_{12}a_{21})k_3| \quad (216)$$

These are exactly the necessary and sufficient conditions expressed in (189), (190) and (191) of result 2.

5.4 Local acceleration properties

At any given (local) configuration **q** in the workspace, the following questions are of theoretical and practical importance.

- Magnitude of the maximum acceleration at any configuration **q** in the workspace
- Magnitude of the isotropic acceleration at any configuration **q** in the workspace

Result 1: The local maximum acceleration $a_{\max,\text{local}}$ of a spatial three degree-of-freedom manipulator at a given configuration **q** is specified by

$$(a_{\max,\text{local}})_{\text{lb}} \leq a_{\max,\text{local}} \leq (a_{\max,\text{local}})_{\text{ub}} \tag{217}$$

where $(a_{\max,\text{local}})_{\text{lb}}$ is given by (188) with $k_1(\mathbf{q},\dot{\mathbf{q}})$, $k_2(\mathbf{q},\dot{\mathbf{q}})$, and $k_3(\mathbf{q},\dot{\mathbf{q}})$ evaluated at that joint variable vector $\dot{\mathbf{q}}$ which maximizes $l(\dot{q}_1,\ \dot{q}_2,\ \dot{q}_3)$ *in equation* (107), and

$$(a_{\max,\text{local}})_{\text{ub}} = a_{\max}(S_{\dot{\mathbf{q}}}) + a_{\max}(S_\tau) \quad . \tag{218}$$

where $a_{\max}(S_{\dot{\mathbf{q}}})$ is given by (107) and $a_{\max}(S_\tau)$ is given by (77).

Result 2: The local isotropic acceleration $a_{\text{iso,local}}$ at a given configuration **q** is specified by

$$\begin{aligned} & a_{\text{iso,local}}(S_{\mathbf{u}}) \\ = \ & \min \begin{bmatrix} \rho(A'B'F'E') - \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_1) \\ \rho(A'D'H'E') - \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_2) \\ \rho(A'B'C'D') - \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_3) \end{bmatrix} \end{aligned} \tag{219}$$

where $\rho(A'B'F'E')$, $\rho(A'D'H'E')$ and $\rho(A'B'C'D')$ are given, respectively, by equations (56) through (58), and where $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_1)$, $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_2)$, and $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_3)$ are given by equation (109).

Proof of result 1:

The local maximum acceleration $a_{\max}$ is the maximum acceleration over all possible state acceleration sets $S_{\mathbf{u}}$ at a given position q in the workspace. Therefore, $a_{\max}$ can be written as

$$a_{\max,\text{local}} = \max(\cup_{\dot{\mathbf{q}}\in F} S_{\mathbf{u}}). \tag{220}$$

It is not possible to find an exact analytical expression for $a_{\max,\text{local}}$. However, we can find an upper bound and lower bound which are very good approximations to $a_{\max,\text{local}}$.

Corresponding to every point P of the set $S_{\dot{\mathbf{q}}}$, we have a state acceleration set $S_{\mathbf{u}}(P)$. Let P' be the furthest point (from the origin) of $S_{\dot{\mathbf{q}}}$, and let $S_{\mathbf{u}}(P')$ be the corresponding state acceleration set. Let the set $S'_{\mathbf{u}}(P')$ obtained by rotating the set $S_{\mathbf{u}}(P')$ about P' till the longest diagonal of $S_{\mathbf{u}}$ is collinear with the line OP' joining the origin to the furthest point P' of $S_{\dot{\mathbf{q}}}$. A lower bound for $a_{\max,\text{local}}$ is given by the distance of the furthest vertex of $S_{\mathbf{u}}$ from the origin, *viz*

$$(a_{\max,\text{local}})_{lb} = \max[d(OA''), d(OB''), d(OC''), d(OD''), d(OE''), d(OF''), d(OG''), d(OH'')], \tag{221}$$

and an upper bound for $a_{\text{iso,local}}$ is given by

$$(a_{\max,\text{local}})_{ub} = d(OP') + d(A''P'), \tag{222}$$

$$(a_{\max,\text{local}})_{\text{ub}} = a_{\max}(S_{\dot{\mathbf{q}}}) + a_{\max}(S_{\tau}). \tag{223}$$

Combining (221) with equation (197) through (204), we obtain equation (188). The values of k_1, k_2 and k_3 in (188) correspond to the furthest vertex P' of $S_{\dot{\mathbf{q}}}$ from the origin, i.e., to that joint variable vector $\dot{\mathbf{q}}$ which maximizes $l(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ in equation (107). This is simply a matter of computing $l(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ at the thirteen vectors defined in subsection 5.2 and determining which of these thirteen vectors maximizes $l(\dot{q}_1, \dot{q}_2, \dot{q}_3)$. This completes the determination of the lower bound $(a_{\max,\text{local}})_{lb}$.

Substituting for $a_{\max}(S_{\dot{\mathbf{q}}})$ and $a_{\max}(S_{\tau})$ from equations (107) and (77), respectively, we obtain equation (218) for the upper bound $(a_{\max,\text{local}})_{lb}$. Thus, Result 1 is proved.

Proof of result 2:

The local isotropic acceleration is obtained in the following steps.

1. The maximum possible isotropic acceleration is obtained when $\dot{\mathbf{q}} = 0$ and is equal to $a_{\text{iso}}(S_{\tau})$ as given by equation (78).

2. Every state acceleration set will have an isotropic acceleration which is less than that given by (78) because the "nonlinearities" effectively reduce the isotropic acceleration. The resulting state isotropic acceleration is $a_{iso}(S_{\mathbf{u}})$ which is given by equation (213).

3. The local isotropic acceleration $a_{\text{iso,local}}$ is the magnitude of the smallest state isotropic acceleration at a given local configuration $\mathbf{q}$, i.e.

$$a_{\text{iso,local}} = \min_{\dot{\mathbf{q}} \in F} a_{\text{iso}}(S_{\mathbf{u}}). \tag{224}$$

4. Using equation (213) and (224), we can express the local isotropic acceleration $a_{\text{iso,local}}$ as

$$\begin{aligned} a_{\text{iso,local}} &= \min_{\dot{\mathbf{q}} \in F} \min[\rho(A'B'F'E') - \rho(K,p_1),\ \rho(A'D'H'E') - \rho(K,p_2),\ \rho(E'F'G'H') - \rho(K,p_3)] \\ &= \min[\min_{\dot{\mathbf{q}} \in F}\{\rho(A'B'F'E') - \rho(K,p_1)\},\ \min_{\dot{\mathbf{q}} \in F}\{\rho(A'D'H'E') - \rho(K,p_2),\ \min_{\dot{\mathbf{q}} \in F}\{\rho(E'F'G'H') - \rho(K,p_3)\}] \end{aligned} \tag{225}$$

5. Since $\rho(A'B'F'E')$, $\rho(A'D'H'E')$ and $\rho(E'F'G'H')$ are constants for a given manipulator and given actuator constraints, (225) can be written as

$$a_{\text{iso,local}} = \min[\rho(A'B'F'E') - \max \rho(K,p_1),\ \rho(A'D'H'E') - \max \rho(K,p_2),\ \rho(E'F'G'H') - \max \rho(K,p_3)]. \tag{226}$$

where $\max[\rho(K,p_1)]$ is the distance from the plane p_1 to the element of $S_{\dot{\mathbf{q}}}$ furthest away from p_1 which we denoted in subsection 5.2 by $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_2)$, $\max[\rho(K,p_2)]$ is the distance from the plane p_2 to the element of $S_{\dot{\mathbf{q}}}$ furthest away from p_2 which we denoted in subsection 5.2 by $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_2)$ and $\max[\rho(K,p_3)]$ is the distance from the plane p_3 to the element of $S_{\dot{\mathbf{q}}}$ furthest away from p_3 which we denoted in subsection 5.2 by $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_3)$. We can therefore write

$$\max \rho(K,p_1) = \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_1) \tag{227}$$

$$\max \rho(K,p_2) = \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_2) \tag{228}$$

$$\max \rho(K,p_3) = \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}),p_3) \tag{229}$$

Combining (226), (227), (228) and (228), we obtain the required result (219). (Note that all quantities in (219) have been analytically determined earlier.)

6 Example:

To demonstrate the ease of applicability of the general acceleration set theory for spatial manipulators developed in the previous sections, we have written simple computer codes to generate the acceleration properties of the various acceleration sets for a common type of 3 d.o.f. spatial manipulator which is shown in Figure 9 and whose o kinematical and dynamical equations are given in the Appendix. (The axis of joint 1 in Figure 9 is vertical). The actual geometric and inertia parameters used in the example are given in Table 3. The dynamical equations have been derived using Kane's dynamical equations (Kane and Levinson 1983; Kane and Levinson 1985; Desa and Roth 1985).

The configuration chosen was $q_1 = o$, $q_2 = 45^\circ$ and $q_3 = 45^\circ$

The joint variable rate ("joint velocity") constraints are

$$\dot{q}_i \leq \dot{q}_{io} = 1 \text{ rad/s} ; \; i = 1,2,3 .$$

The torque constraints are

$$\tau_i \leq \tau_{io} \; , \quad i = 1,2,3 .$$

τ_{io} may be thought of as the size (or maximum torque rating) of the actuators; the numerical values of τ_{10}, (i = 1,2,3), are given in Table 3.

The properties of the state acceleration set were computed at $q_1 = o$, $q_2 = 45^\circ$ and $q_3 = 45^\circ$; $\dot{q}_1 = 1$ rad/s, $\dot{q}_2 = 1$ rad/s $\dot{q}_3 = -1$ rad/s

In order to show how the theory might be used for design purposes we have determined the acceleration properties for three cases (Table 4). Five acceleration properties have been determined in each case: the maximum and isotropic acceleration of the set S_τ, the maximum and isotropic accleration of the state acceleration set and the (local) isotropic acceleration at the configuration $(0, 45^\circ, 45^\circ)^T$.

In all three cases the sizes of the first two actuators remain constant ($\tau_{10} = 35$ N-m and $\tau_{20} = 8.2$ Nm) and the size of the third actuator (driving link 3) is varied. In Case 1 of Table 4 ($\tau_{30} = 0.17$ N-m), the end-effector does not have either a state or local isotropic acceleration). When the size of actuator 3 is increased to 0.4 N-m (Case 2), we obtain a state isotropic acceleration of 0.93 m/s^2 but the local isotropic acceleration is very small 0.03 m/s^2. Therefore for given τ_{10} and τ_{20} , τ_{30} must be greater than 0.4 N-m in order that we may have a local isotropic acceleration at the specified configuration q. Case 3 shows that for

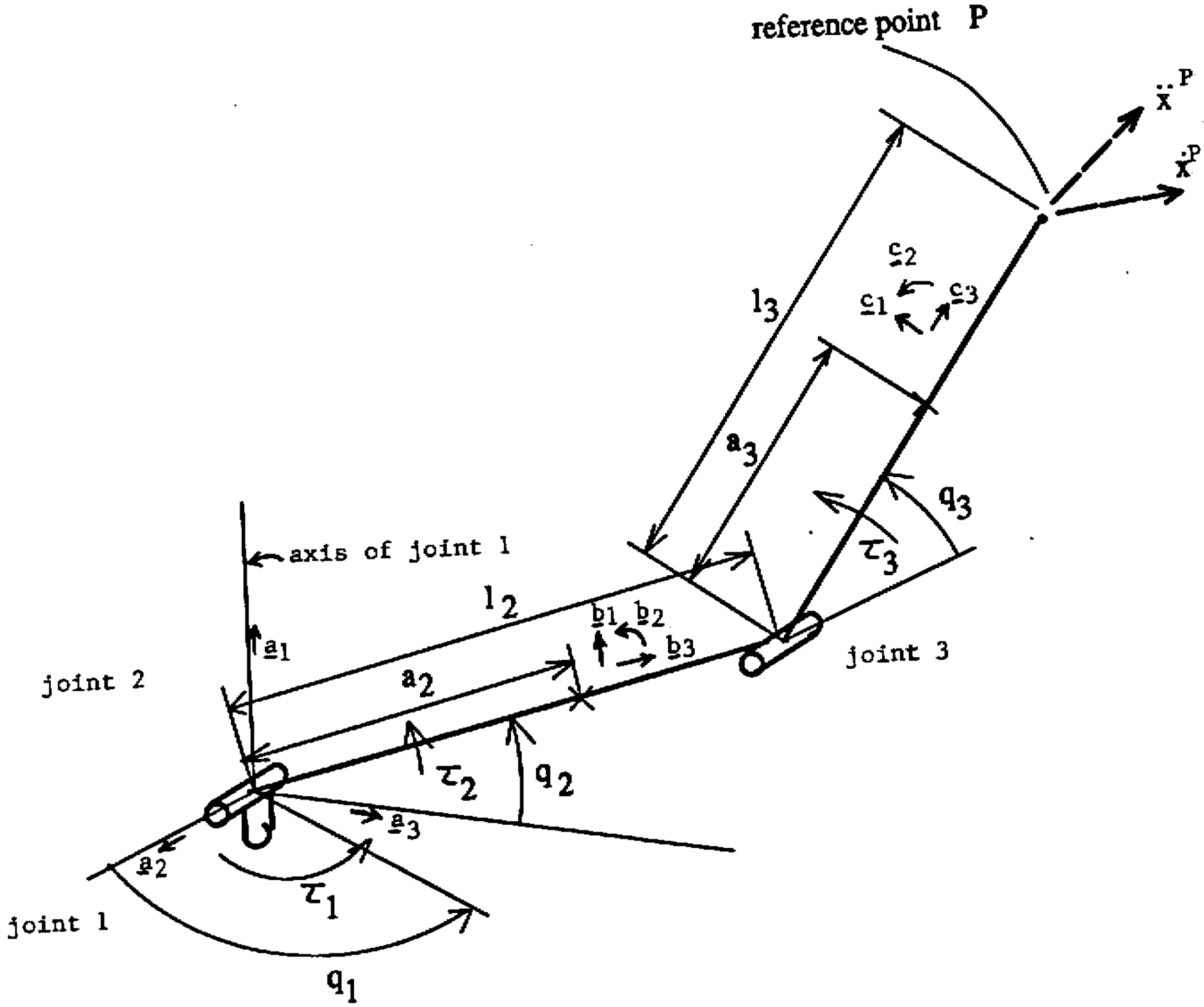

Figure 9: Schematic diagram of a three degree-of-freedom manipulator

actuator size τ_{30} of 0.6 N-m we have a local isotropic acceleration of 1.61 m/s^2. The designer must then decide (from past experience) whether this magnitude of isotropic acceleration is reasonable.

Comments:

1. These computations can be repeated for various configurations in the workspace after which decisions can be made regarding actuator sizes.

2. Algorithms for the determination of minimum actuator sizes to achieve a desired isotropic acceleration are given in (Desa and Kim 1989-2) for the planar case. The extension to the spatial case is relatively straightforward.

ℓ_1 = 0.0	ℓ_2 = 0.303	ℓ_3 = 0.254	(m)
a_1 = 0.0	a_2 = 0.196	a_3 = 0.094	(m)
m_1 = 3.5	m_2 = 2.259	m_3 = 1.129	(kg)
I_1 = 1.2	J_1 = ------	K_1 = ------	(kg-m^2)
I_2 = .129	J_2 = .129	$K_2 \cong 0$	(kg-m^2)
I_3 = .003	J_3 = .003	$K_3 \cong 0$	(kg-m^2)

Table 3: Parameters for the spatial manipulator (see Figure 9 and the Appendix).

	Actuator Torques			Acceleration			Properties	
Case	τ_{10} (N-m)	τ_{20} (N-m)	τ_{30} (N-m)	$A_{max}(s_\tau)$ (m/s^2)	$A_{iso}(s_\tau)$ (m/s^2)	$A_{max}(s_u)$ (m/s^2)	$A_{iso}(s_u)$ (m/s^2)	$A_{iso,\ local}$ (m/s^2)
1.	35	8.2	0.17	20.3	1.35	23.7	0	0
2.	35	8.2	0.4	25.06	3.16	29.1	0.93	0.03
3.	35	8.2	0.6	30.3	4.75	33.9	2.51	1.61

Table 4: Acceleration Properties for the manipulator of Section 6.

7 Summary and Conclusions:

In this paper, we extended the acceleration set theory for planar manipulators, developed in (Desa and Kim, 1989-1), to spatial manipulators. As in the planar case we have accomplished the following:

- Given the kinematical and dynamical equations of a manipulator, we have defined the image set S_τ corresponding to the set T of actuator torques, and the image set $S\dot{q}$ corresponding to the set F of the joint variable rates. We have also defined the state acceleration set S_u at a specified point u in the state space.
- We have determined the image sets, S_τ and $S\dot{q}$, and the state acceleration set S_u.
- We have characterized the image sets S_τ and the state acceleration set S_u by their maximum and isotropic acceleration. The image set $S\dot{q}$ has been also characterized by its maximum acceleration.
- At a configuration or position, **q**, in the workspace, we have established two local acceleration properties: the local maximum acceleration and the local isotropic acceleration. The local maximum acceleration specifies the magnitude of the maximum acceleration of (a reference point on) the end-effector. The local isotropic acceleration specifies the magnitude of the maximum available acceleration of the end-effector in all directions.

We then demonstrated the application of the acceleration set theory for spatial manipulator to the 3 d.o.f. spatial manipulator shown in Figure 9.

We have, therefore, demonstrated the hypothesis which we stated in the introduction, i.e., that the analytical properties of acceleration sets can be determined from the properties of the linear and quadratic mappings which define them (the acceleration sets). Furthermore, the acceleration properties of interest - especially the isotropic acceleration - have been determined in terms of the manipulator parameters, the torque limits and joint variable rate ("joint velocity") limits. These results can therefore be applied to manipulator design problems as demonstrated in (Desa and Kim, 1989-2).

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Appendix: Functional relationships for the spatial 3 d.o.f. manipulator of Figure 9.

Notation: (See Figure 9)

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$:	dextral orthogonal set of unit vectors fixed in link 1 and parallel to the central principal moments of inertia of link 1.
$\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$:	dextral orthogonal set of unit vectors fixed in link 2 and parallel to the central principal moments of inertia of link 2.
$\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$:	dextral orthogonal set of unit vectors fixed in link 3 and parallel to the central principal moments of inertia of link 3.
ℓ_1 :	length of link 2
ℓ_3 :	length of link 3
a_2 :	distance from joint axis of link 2 to center of mass of link 2
a_3 :	distance from joint axis of link 3 to center of mass of link 3
m_1 :	mass of link 1
m_2 :	mass of link 2
m_3 :	mass of link 3
I_1, J_1, K_1 :	central principal moments of inertia of link 1 for axes parallel to $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively.[1]
I_2, J_2, K_2 :	central principal moments of inertia of link 2 for axes parallel to $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ respectively.
I_3, J_3, K_3 :	central principal moments of inertia of link 3 for axes parallel to $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ respectively.

(The input and output variables are as defined in section 3.1)

1. Jacobian matrix

The joint velocity is related to the velocity $\dot{x}$ of the point P in Cartesian space by the relation $\dot{x} = J\dot{q}$

The Jacobian matrix **J** for a spatial three degree-of-freedom manipulator in Figure 9 is the following:

$$\mathbf{J} = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ 0 & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix}$$

where

$$j_{11} = \sin q_1 (l_2 \cos q_2 + l_3 \cos(q_2 + q_3))$$

$$j_{12} = -\cos q_1 (l_2 \sin q_2 + l_3 \sin(q_2 + q_3))$$

$$j_{13} = -l_3 \cos q_1 \sin(q_2 + q_3)$$

$$j_{22} = l_2 \cos q_2 + l_3 \cos(q_2 + q_3)$$

$$j_{23} = l_3 \cos(q_2 + q_3)$$

[1] For link 1, since the first joint axis is parallel to $\mathbf{a}_1$, only the principal moment I_1 is of importance in the dynamic equations.

$$j_{31} = -\cos q_1 (l_2 \cos q_2 + l_3 \cos(q_2 + q_3))$$

$$j_{32} = -\sin q_1 (l_2 \sin q_2 + l_3 \sin(q_2 + q_3))$$

$$j_{33} = -\sin q_1 l_3 \sin(q_2 + q_3)$$

When the above relation is differentiated with respect to the time, we obtain the following equation,

$$\ddot{\mathbf{x}} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = \mathbf{J}\ddot{\mathbf{q}} - \mathbf{F} < \dot{\mathbf{q}} >^2 - \mathbf{G}[\dot{\mathbf{q}}]^2 \tag{230}$$

where **F**, **G** are matrices with the following elements:

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{31} & f_{32} & f_{33} \end{bmatrix} ,$$

where

$$f_{13} = \cos q_1 l_3 \cos(q_2 + q_3)$$

$$f_{22} = l_2 \sin q_2 + l_3 \sin(q_2 + q_3)$$

$$f_{31} = -\sin q_1 (l_2 \cos q_2 + l_3 \cos(q_2 + q_3))$$

$$f_{32} = \sin q_1 (l_2 \cos q_2 + l_3 \cos(q_2 + q_3))$$

$$f_{33} = \sin q_1 l_3 \cos(q_2 + q_3) \quad ,$$

and

$$\mathbf{G} = \begin{bmatrix} 0 & g_{12} & 0 \\ 0 & g_{22} & 0 \\ 0 & g_{32} & 0 \end{bmatrix} ,$$

where

$$g_{12} = \cos q_1 l_3 \cos(q_2 + q_3)$$

$$g_{22} = l_3 \sin(q_2 + q_3)$$

$$g_{23} = \sin q_1 l_3 \cos(q_2 + q_3)$$

2. Dynamic equation

The dynamic behavior of the manipulator is described by the following equation:

$$\mathbf{D}\ddot{\mathbf{q}} + \mathbf{U} < \dot{\mathbf{q}} >^2 + \mathbf{W}[\dot{\mathbf{q}}]^2 + \mathbf{p} = \tau. \tag{231}$$

The components of matrices **D**, **U**, and **W** are as follows:

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix}$$

where

$$\begin{aligned}
d_{11} &= \mathrm{I}_1 + (I_2 + m_2 a_2^2)\cos^2 q_2 + I_3 \cos^2(q_2 + q_3) + m_3(l_2 \cos q_2 + a_3 \cos(q_2 + q_3))^2 \\
d_{22} &= J_2 + m_2 a_2^2 + J_3 + m_3(a_3^2 + 2a_3 l_2 \cos q_3 + l_2^2) \\
d_{23} &= J_3 + m_3(a_3^2 + a_3 l_2 \cos q_3) \\
d_{32} &= d_{23} \\
d_{33} &= J_3 + m_3 a_3^2
\end{aligned}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 \\ u_{21} & 0 & u_{23} \\ u_{31} & u_{32} & 0 \end{bmatrix}$$

where

$$\begin{aligned}
u_{21} &= (I_2 + m_2 a_2^2)\cos q_2 \sin q_2 + I_3 \cos(q_2 + q_3)\sin(q_2 + q_3) + \\
&+ m_3(l_2 \cos q_2 + a_3 \cos(q_2 + q_3))(l_2 \sin q_2 + a_3 \sin(q_2 + q_3)) \\
u_{23} &= m_3 l_2 a_3 \sin q_3 \\
u_{31} &= I_3 \cos(q_2 + q_3)\sin(q_2 + q_3) + m_3(l_2 \cos q_2 + a_3 \cos(q_2 + q_3)) a_3 \sin(q_2 + q_3) \\
u_{32} &= u_{23}
\end{aligned}$$

$$\mathbf{W} = \begin{bmatrix} w_{11} & 0 & w_{13} \\ 0 & w_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where

$$
\begin{aligned}
w_{11} &= -[(I_2 + m_2 a_2^2)\cos q_2 \sin q_2 + I_3 \cos(q_2+q_3)\sin(q_2+q_3) + \\
&\quad + m_3(l_2 \cos q_2 + a_3 \cos(q_2+q_3))(l_2 \sin q_2 + a_3 \sin(q_2+q_3))] \\
w_{13} &= -[I_3 \cos(q_2+q_3)\sin(q_2+q_3) + m_3(l_2 \cos q_2 + a_3 \cos(q_2+q_3))a_3 \sin(q_2+q_3)] \\
w_{22} &= m_3 l_2 a_3 \sin q_3
\end{aligned}
$$

The nonlinear vectors, $< \dot{\mathbf{q}} >^2$ and $[\dot{\mathbf{q}}]^2$ are as follows:

$$
< \dot{\mathbf{q}} >^2 = \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \dot{q}_3^2 \end{bmatrix}
$$

$$
[\dot{\mathbf{q}}]^2 = \begin{bmatrix} 2\dot{q}_1\dot{q}_2 \\ 2\dot{q}_2\dot{q}_3 \\ 2\dot{q}_3\dot{q}_1 \end{bmatrix}
$$

$$
\mathbf{p} = \begin{bmatrix} 0 \\ p_2 \\ p_3 \end{bmatrix}
$$

where

$$
\begin{aligned}
p_2 &= [m_2 a_2 \cos q_2 + m_3(l_2 \cos q_2 + a_3 \cos(q_2+q_3))]g \\
p_3 &= m_3(l_2 \cos q_2 + a_3 \cos(q_2+q_3))g
\end{aligned}
$$

3. Acceleration equation

The expression for the acceleration of the end-effector is as follows:

$$
\ddot{\mathbf{x}} = \mathbf{A}\tau + \mathbf{B} < \dot{\mathbf{q}} >^2 + \mathbf{N}[\dot{\mathbf{q}}]^2 + \mathbf{s} \tag{232}
$$

where

$$
\begin{aligned}
\mathbf{A} &= \mathbf{J}\mathbf{D}^{-1} \\
\mathbf{B} &= -\mathbf{A}\mathbf{U} + \mathbf{F} \\
\mathbf{M} &= -\mathbf{A}\mathbf{W} + \mathbf{G} \\
\mathbf{s} &= -\mathbf{A}\mathbf{p}
\end{aligned}
\tag{233}
$$

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