Acceleration Sets of Planar Manipulators
Part I: Theory

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Abstract

This report develops a systematic approach for determining the acceleration capability and the acceleration properties of the end-effector of a planar two degree-of-freedom manipulator. The acceleration of the end-effector at a given configuration of the manipulator is a linear function of the actuator torques and a (nonlinear) quadratic function of the "joint-velocities". By decomposing the functional relationships between the inputs (actuator torques and "joint-velocities") and the output (acceleration of the end-effector) into two fundamental mappings, a linear mapping between the actuator torque space and the acceleration space of the end-effector and a quadratic (nonlinear) mapping between the "joint-velocity" space and the acceleration space of the end-effector, and by deriving the properties of these two mappings, it is possible to determine the properties of all acceleration sets which are the images of the appropriate input sets under the two fundamental mappings. The determination of the properties of the quadratic mapping, a key feature of the present work, allows us to obtain analytic expressions relating important acceleration properties of the end-effector to all the manipulator parameters and input variables of interest.

1 Introduction

In this paper, we develop and apply a systematic approach for studying the acceleration capability and acceleration properties of (a reference point on) the end-effector of a planar two degree-of-freedom manipulator. The application of the theory developed in this paper to two important problems which arise in the design of manipulators -selection of a manipulator type and determination of actuator sizes - are described in companion paper (Desa and Kim, 1989). Acceleration theory for spatial manipulators is developed in a third paper (Kim and Desa, 1989).

An informal statement of the acceleration problem is as follows:

Consider the planar two degree-of-freedom manipulator shown schematically in Figure 1. We are interested in studying the acceleration of a reference point P on link 2. (P is typically a point on the joint axis of the end-effector: therefore the acceleration of P is often loosely referred to as the acceleration of the end-effector.) The usefulness of studying the acceleration of the end-effector of manipulators has been discussed in (Yoshikawa, 1985), (Khatib and Burdick, 1987) and (Graettinger and Krogh, 1988) and will additionally be demonstrated in (Desa and Kim, 1989).

As will be shown below, the acceleration capability of the point P under various conditions is best described by certain acceleration sets. Two properties which are used, in general, to characterize these sets are the maximum possible magnitude of the acceleration of P and the maximum magnitude of the acceleration of P which is available in all directions. The former property is simply called the maximum acceleration of P and the latter the isotropic acceleration of P (Khatib and Burdick, 1987).

The study of the acceleration properties of the "end-effector" has been a subject of recent interest (Yoshikawa, 1985; Khatib and Burdick, 1987; Graettinger and Krogh, 1988). It is therefore useful to clearly state what makes the problem of studying acceleration properties complex and how these researchers have addressed this complexity.

The acceleration of the reference point P at a given configuration (in the workspace of the manipulator) is a linear function of the actuator torques and a (nonlinear) quadratic function of the rates of changes of the joint-variables ("joint velocities"). The complexity of the "acceleration problem" arises from these quadratic nonlinearities in the "joint velocities". (Yoshikawa, 1985) studied the acceleration of (a reference point P on) the end-effector in connection with developing a dynamic manipulability measure:

in this study the nonlinearities were essntially ignored since the measure was estimated at zero "joint velocities". In studying isotropic acceleration, (Khatib and Burdick, 1987) dealt with the nonlinearities in a somewhat ad-hoc fashion by evaluating isotropic acceleration at a "low" and a "high" joint velocity vector. (Graettinger and Krogh, 1988) handled the nonlinearities by posing the problem of determining the isotropic acceleration as an optimization problem.

In contrast to the above approaches, the present paper demonstrates how these nonlinearities can be handled in an analytical manner. The fundamental hypothesis of this paper is the following: By decomposing the functional relationships between the inputs (actuator torques and joint variable rates) and the output (acceleration of P) into two fundamental mappings, a linear mapping between actuator torque space and the acceleration space of point P and a quadratic (nonlinear) mapping between the "joint velocity" space and the acceleration space of P, and by deriving the properties of these two mappings, it is possible to determine the properties of all acceleration sets which are the images of the appropriate input sets under the two fundamental mappings.

The properties of linear mappings are well-known. The determination of the properties of the quadratic mapping between the joint velocities and the acceleration-space of P is one of the contributions of this paper and permits us to obtain exact analytic solutions for the isotropic acceleration under various conditions.

In summary, the contributions of this paper are the following:

1. Development of a systematic approach (stated in section 2) for defining, determining and characterizing acceleration sets.

2. Closed-form analytic expressions relating important acceleration properties of manipulators to all the manipulator parameters and input variables (torques, joint variable rates or "joint velocities") of interest. (The only exception is the maximum local acceleration which is specified in terms of tight lower and upper bounds in section 6.)

3. Necessary and sufficient conditions for the existence of isotropic acceleration. (Earlier studies seem to implicitly assume that isotropic acceleration always exists.) These conditions are stated explicitly in terms of manipulator parameters and input variables.

4. Analytical expressions for determining the maximum and isotropic acceleration of the end-effector at any ("local") configuration of the manipulator.

5. The theory treats nonlinearities in an "exact" manner (as mentioned above).

One consequence of 2 and 3 above is the development of simple algorithms (Desa and Kim 1989) for sizing actuators in order to guarantee a specified isotropic acceleration. The theory developed in this paper is also applicable to two degree-of-freedom manipulators with closed-chains (Desa and Kim, 1989).

The next section, which describes our approach, also provides the dual function of being a "road-map" of the paper.

2 Description of the approach

A systematic approach for studying the acceleration of (a reference point P on) the end-effector based on the use of input-output mappings is as follows:

1. Define the input variables and output variables of interest (subsection 3.1). The output of interest is the acceleration of the reference point P.
2. Define the input sets of interest (subsection 3.1).
3. Define the input-output functional relations. These are obtained from the dynamical and kinematical equations of the manipulator (subsection 3.2).
4. Define fundamental mappings from these functional relations (subsection 3.3). There are two fundamental mappings, a linear mapping and a quadratic mapping.
5. Define the image sets of the input sets under the mappings obtained in set 4 (subsection 3.4). These image sets are the acceleration sets of interest.
6. Define general properties which can be used to characterize ("measure") acceleration sets (subsection 3.5).
7. Determine the properties of the mappings defined in step 4 (section 4).
8. Determine the acceleration sets defined in step 5 using the properties of the mappings obtained in step 7 (section 4).
9. Determine the specific properties of the acceleration sets determined in step 8 using the "measures" or general properties defined in step 6 (section 5).
10. Determine the local acceleration properties for any configuration $\mathbf{q}$ of the manipulator using the properties of the acceleration sets obtained in step 9 (section 6).

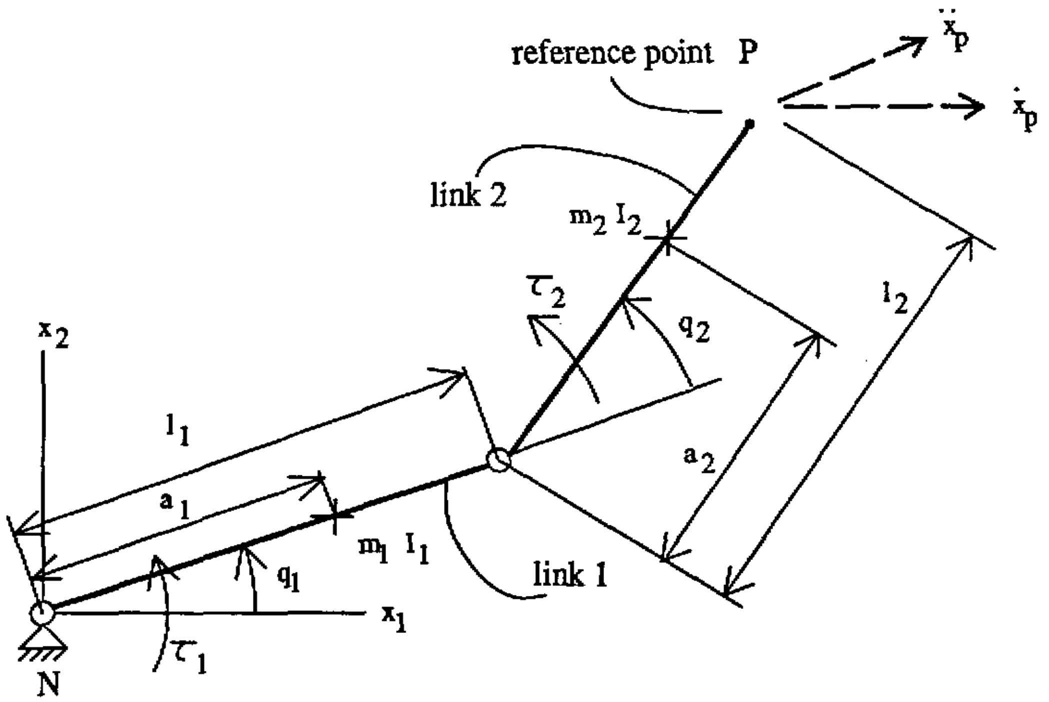

Figure 1: Schematic diagram of a planar two degree-of-freedom manipulator

3 Definition of the acceleration sets

3.1 Manipulator input and output variables

Consider a serial two degree-of-freedom manipulator with two revolute joints shown in Figure 1. In this subsection, we define the link parameters, the input variables, the input sets, the output variables and the output sets for a planar two degree-of-freedom manipulator. The manipulator is assumed to be rigid with negligible joint friction and operates in a horizontal plane perpendicular to the "gravity vector". (The case of manipulators operating in gravity fields is relatively straightforward and is dealt with in (Kim and Desa, 1989).)

The link parameters necessary for describing the kinematic and dynamic behavior of the planar two degree-of-freedom manipulator (Figure 1) are as follows. Let l_1 denote the length of link 1, a_1 the distance from joint axis 1 to the center of mass of link 1, m_1 the mass of link 1, and I_1 the principal moment of inertia of link 1 with respect to its center of mass about an axis perpendicular to the plane of the motion.

Similarly, let l_2, a_2, m_2, and I_2 denote the corresponding variables for link 2 (see Figure 1).

Next, we define the input variables, the input constraints and the corresponding input sets of the two degree-of-freedom manipulator. Let q_1 and q_2 denote the generalized coordinates of the manipulator (see Figure 1), q_1 being the joint variable at joint 1 and q_2 the joint variable at joint 2. Define

$$\mathbf{q} \triangleq \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{1.1}$$

to be the vector of joint variables; the corresponding vector space of all $\mathbf{q}$ is called the joint space. If

$$q_{iL} \leq q_i \leq q_{iU}, \ i = 1,2 \tag{1.2}$$

denotes the constraint on joint variable i, then we can define the workspace W of a manipulator as

$$W = \{\mathbf{q} | q_{iL} \leq q_i \leq q_{iU}, \ i = 1,2\}. \tag{1.3}$$

Let $\dot{q}_1$ and $\dot{q}_2$ denote, respectively, the rates of change of the joint variables q_1 and q_2; $\dot{q}_1$ and $\dot{q}_2$ will be referred to as joint variable rates for short. Define

$$\dot{\mathbf{q}} \triangleq \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{1.4}$$

to be the vector of the joint variable rates. If

$$| \dot{q}_i | \leq \dot{q}_{io}, \ i = 1,2 \tag{1.5}$$

denotes the constraints on the joint variable rates, then we can define

$$F = \{\dot{\mathbf{q}} | \ | \dot{q}_i | \leq \dot{q}_{io}, \ i = 1,2\} \tag{1.6}$$

to be the set of all the possible joint variable rate vectors; graphically F can be represented by (the interior and boundary of) the rectangle $J_1K_1J_2K_2$ shown in Figure 2.

Let τ_1 and τ_2 denote the actuator torques, respectively, at joints 1 and 2, and define

$$\tau \triangleq \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \tag{1.7}$$

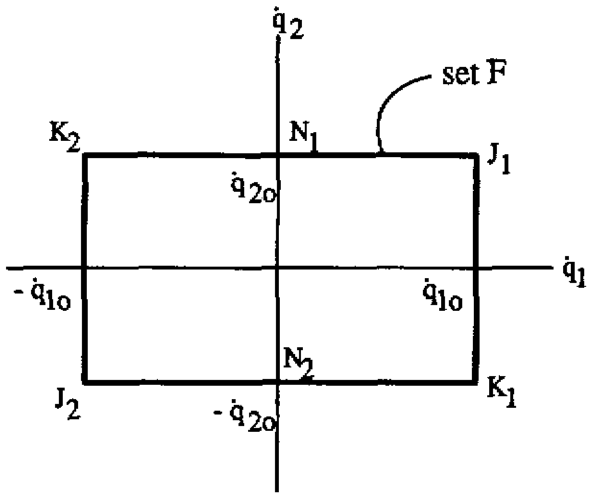

Figure 2: Set of the joint variable rates of a two degree-of-freedom manipulator

to be the actuator torque vectors.[1] Let

$$|\tau_i| \leq \tau_{io}, \; i = 1,2 \tag{1.8}$$

denote the constraints on the actuator torques at joints 1 and 2. We define

$$T = \{\tau \,|\, |\tau_i| \leq \tau_{io}, \; i = 1,2\} \tag{1.9}$$

to be the set of the allowable actuator torques; graphically T can be represented by (the interior and boundary of) the rectangle $ABCD$ in Figure 3.

The vectors $\mathbf{q}$, $\dot{\mathbf{q}}$ and τ will be referred to as the input variables (more precisely the input variable vectors) of the manipulator. We will also refer to the vector $\mathbf{q}$ as a configuration of the manipulator.

Let (x_1, x_2) denote the coordinates of a reference point P on link 2 (see Figure 1) in a coordinate system fixed to the base reference frame N; (x_1, x_2) are commonly referred to as task coordinates. Define

$$\mathbf{x}^p \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1.10}$$

to be the vector of task coordinates; the corresponding vector space of all $\mathbf{x}^p$ is called the task space.

[1]The vectors of actuator torques, joint variables, and joint variable rates denote column matrices, not physical vectors.

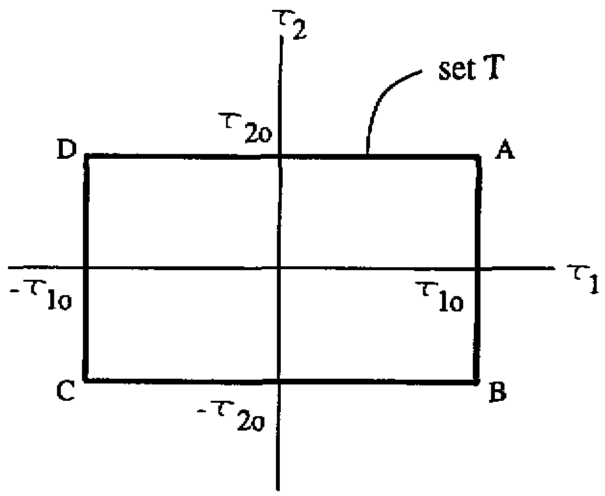

Figure 3: Set of the actuator torques of a two degree-of-freedom manipulator

The velocity $\dot{\mathbf{x}}^P$ and the acceleration $\ddot{\mathbf{x}}^P$ of the point P of the manipulator are, respectively, given by

$$\dot{\mathbf{x}}^P = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \tag{1.11}$$

and

$$\ddot{\mathbf{x}}^P = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}. \tag{1.12}$$

The acceleration of P, $\ddot{\mathbf{x}}^P$, is the output variable of interest in the present work. The corresponding vector space A of all possible $\ddot{\mathbf{x}}^P$ is called the acceleration space, expressed by

$$A = \{\ddot{\mathbf{x}} \mid \ddot{\mathbf{x}} \in R^2\}. \tag{1.13}$$

3.2 Functional relations between the inputs $\dot{\mathbf{q}}$, τ and the acceleration $\ddot{\mathbf{x}}^P$

The next step is to obtain the functional relations between the acceleration $\ddot{\mathbf{x}}^P$ and the inputs $\dot{\mathbf{q}}$ and τ for a given configuration $\mathbf{q}$. In this subsection, we show how the necessary functional relations can be obtained from the manipulator dynamic equations and the (so-called) manipulator Jacobian relation.

The dynamic behavior of the two degree-of-freedom planar manipulator in the joint space can be obtained using well-known methods (Kane and Levinson, 1983; Kane and Levinson, 1985; Desa and

Roth, 1985) and is described by the following pair of equations:

$$d_{11}\ddot{q}_1 \quad + \quad d_{12}\ddot{q}_2 - w_{12}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) = \tau_1, \tag{1.14}$$

$$d_{21}\ddot{q}_1 \quad + \quad d_{22}\ddot{q}_2 + w_{21}\dot{q}_1^2 = \tau_2, \tag{1.15}$$

where the coefficients, d_{ij} $(i,j = 1,2)$ and v, are given in the Appendix.

Defining the following matrix operators

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}, \tag{1.16}$$

$$\mathbf{W} = \begin{bmatrix} 0 & w_{12} \\ w_{21} & 0 \end{bmatrix}, \tag{1.17}$$

$$\ddot{\mathbf{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}, \tag{1.18}$$

$$\{\dot{\mathbf{q}}\}^2 = \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \end{bmatrix}, \tag{1.19}$$

dynamic equations,(1.14) and (1.15), become

$$\mathbf{D}\ddot{\mathbf{q}} + \mathbf{W}\{\dot{\mathbf{q}}\}^2 = \tau. \tag{1.20}$$

Note that equation (1.20) is the most general expression of the dynamics of a two degree-of-freedom planar manipulator. The matrices **D** and **W** standard for various planar manipulator types are given in the Appendix. The matrix **D** is the mass matrix of the manipulator.

Since the matrix **D** is always invertible, we can write (1.20) in a more convenient form for our purposes as

$$\ddot{\mathbf{q}} = \mathbf{D}^{-1}[\tau - \mathbf{W}\{\dot{\mathbf{q}}\}^2]. \tag{1.21}$$

A crucial step in the acceleration analysis of a two degree-of-freedom manipulator is the definition of the matrix operator **W** and $\{\dot{\mathbf{q}}\}^2$, which allows all the "non-linearities" (i.e. terms in the dynamic equations (1.14) and (1.15) which are non-linear in the joint variable rates, $\dot{q}_1$ and $\dot{q}_2$) to be written as the product of **W** and $\{\dot{\mathbf{q}}\}^2$. The notation $\{\}^2$ is used to draw attention to the fact that the elements of the vector $\{\dot{\mathbf{q}}\}^2$ are quadratic in the joint variable rates $\dot{q}_1$ and $\dot{q}_2$. Note that $\{\dot{\mathbf{q}}\}^2$ is a vector and should not be confused with the scalar $\dot{\mathbf{q}}^2$ which is the square of the magnitude of $\dot{\mathbf{q}}$.

The relation between the velocity, $\dot{\mathbf{x}}^p$, of the point P, and the joint variable rate vector $\dot{\mathbf{q}}$ is well known (Desa and Roth, 1985):

$$\dot{\mathbf{x}}^p = \mathbf{J}\dot{\mathbf{q}} \tag{1.22}$$

where $\mathbf{J}$ is a (2×2) matrix called the manipulator Jacobian. The detailed expressions of the Jacobian matrix for various planar manipulator types are given in the Appendix.

To obtain the expression for the acceleration $\ddot{\mathbf{x}}^p$ of the point P, we differentiate equation (1.22),

$$\ddot{\mathbf{x}}^p = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}. \tag{1.23}$$

In the Appendix, we show that the second term in (1.23), $\dot{\mathbf{J}}\dot{\mathbf{q}}$, can be written in the form

$$\dot{\mathbf{J}}\dot{\mathbf{q}} = -\mathbf{E}\{\dot{\mathbf{q}}\}^2 \tag{1.24}$$

where matrix $\mathbf{E}$ is skew-symmetric.

Substituting equation (1.24) into (1.23), we obtain

$$\ddot{\mathbf{x}}^p = \mathbf{J}\ddot{\mathbf{q}} - \mathbf{E}\{\dot{\mathbf{q}}\}^2. \tag{1.25}$$

Defining the quantities,

$$\mathbf{A} = \mathbf{J}\mathbf{D}^{-1}, \tag{1.26}$$

$$\mathbf{B} = -\mathbf{A}\mathbf{W} - \mathbf{E}, \tag{1.27}$$

it is easy to verify that the expression for the acceleration $\ddot{\mathbf{x}}^p$ of the point P, obtained by combining equation (1.20) with equations (1.25) through (1.27), is given by

$$\ddot{\mathbf{x}}^p = \mathbf{A}\tau + \mathbf{B}\{\dot{\mathbf{q}}\}^2 \tag{1.28}$$

where $\mathbf{A}, \mathbf{B}$ are configuration dependent.

Equation (1.28) expresses the required (Input-Output) functional relation between the input variables, $\dot{\mathbf{q}}$ and τ, and the acceleration $\ddot{\mathbf{x}}^p$ of the point P (the output variable) at a given configuration q. It is important to note that the definition of the matrix "operators" $\mathbf{W}$, $\mathbf{E}$ and $\{\dot{\mathbf{q}}\}^2$ enables us to write the dynamic equations in the compact form (1.28) which is critical in the sequel.

3.3 Mappings

In this subsection, we define two fundamental mappings between the input variables and the acceleration $\ddot{\mathbf{x}}^P$ of the point P (the output variable).

It is convenient to regard the functional relation (1.28) as a mapping between the input variables $\dot{\mathbf{q}}$ and τ and the output variable $\ddot{\mathbf{x}}^P$ for a given configuration $\mathbf{q}$ of the manipulator. Furthermore, defining

$$\alpha_\tau \triangleq \begin{bmatrix} \alpha_{1\tau} \\ \alpha_{2\tau} \end{bmatrix} \triangleq \mathbf{A}\tau \tag{1.29}$$

and

$$\alpha_{\dot{\mathbf{q}}} \triangleq \begin{bmatrix} \alpha_{1\dot{\mathbf{q}}} \\ \alpha_{2\dot{\mathbf{q}}} \end{bmatrix} \triangleq \mathbf{B}\{\dot{\mathbf{q}}\}^2, \tag{1.30}$$

equation (1.28) can be written as

$$\ddot{\mathbf{x}}^P = \alpha_\tau + \alpha_{\dot{\mathbf{q}}}. \tag{1.31}$$

The following two simple and obvious relations are useful when we define acceleration sets below:

$$\ddot{\mathbf{x}}^P(\dot{\mathbf{q}} = 0) \quad = \quad \alpha_\tau = \mathbf{A}\tau \tag{1.32}$$

$$\ddot{\mathbf{x}}^P(\tau = 0) \quad = \quad \alpha_{\dot{\mathbf{q}}} = \mathbf{B}\{\dot{\mathbf{q}}\}^2. \tag{1.33}$$

It is convenient to think of the vector α_τ as the contribution of the torques to the acceleration of the reference point P, and the vector $\alpha_{\dot{\mathbf{q}}}$ as the contribution of the joint variable rates to the acceleration of P. The sum of these two vectors, therefore, gives us the acceleration of P as expressed by equation (1.31) for a two degree-of-freedom manipulator.

Equation (1.29) can be viewed as a linear, configuration-dependent, mapping between the torque vector τ and its contribution α_τ to the acceleration of P. Similarly, equation (1.30) can be viewed as a quadratic, configuration-dependent, mapping between the joint variable rate vector $\dot{\mathbf{q}}$ and its contribution $\alpha_{\dot{\mathbf{q}}}$ to the acceleration of P. These are the two mappings of interest in this section.

3.4 Manipulator acceleration sets

Having defined two fundamental mappings of interest, we are interested in the image sets of the input sets under the mappings (1.29) and (1.30) at a given configuration $\mathbf{q}$ of the manipulator. There are three image sets of interest.

3.4.1 Image set S_τ of the actuator torque set T under the linear mapping

For a given set T of the actuator torques τ described by equation (1.9), and represented graphically by the rectangle $ABCD$ in the τ - plane (see Figure 3), we define the image set S_τ of T under the linear mapping (1.32) as

$$S_\tau = \{\ddot{\mathbf{x}}^p | \ddot{\mathbf{x}}^p(\dot{\mathbf{q}} = 0) = \mathbf{A}\tau, \tau \in T\}. \tag{1.34}$$

(Note that S_τ lies in the acceleration plane A.) From equation (1.32) and the above definition (1.34), we see that S_τ represents the set of all possible accelerations (the acceleration capability of the manipulator) when it is at rest ($\dot{\mathbf{q}} = 0$) in any configuration $\mathbf{q}$ and the actuators are turned on.

3.4.2 Image set $S_{\dot{\mathbf{q}}}$ of the joint variable rate set F under the quadratic mapping

For a given set F of the joint variable rates $\dot{\mathbf{q}}$ described by equation (1.6), and represented graphically by the rectangle $J_1K_1J_2K_2$ in the $\dot{\mathbf{q}}$ - plane (see Figure 2), we define the image set $S_{\dot{\mathbf{q}}}$ of F under the quadratic mapping (1.33) as

$$S_{\dot{\mathbf{q}}} = \{\ddot{\mathbf{x}}^p | \ddot{\mathbf{x}}^p(\tau = 0) = \mathbf{B}\{\dot{\mathbf{q}}\}^2, \dot{\mathbf{q}} \in F\}. \tag{1.35}$$

(Note that $S_{\dot{\mathbf{q}}}$ lies in the acceleration plane A.) From equation (1.33) and the above definition (1.35), we see that the image set $S_{\dot{\mathbf{q}}}$ represents the set of all possible accelerations (the acceleration capability of the manipulator) when the actuators are turned off ($\tau = 0$) in any configuration $\mathbf{q}$.

3.4.3 State acceleration set

When a manipulator is in motion, the (dynamic) state of a manipulator can be specified by the joint variables, (q_1, q_2), and joint variable rates, ($\dot{q}_1$, $\dot{q}_2$). The state vector $\mathbf{u}$ which characterizes the dynamic state of the manipulator is defined as follows:

$$\mathbf{u} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}. \tag{1.36}$$

For a specified dynamic state of a two degree-of-freedom manipulator, the second term of the acceleration $\ddot{\mathbf{x}}^P$ in equation (1.28) is a constant vector, which we denote by $\mathbf{k}(\mathbf{u})$ and define as follows:

$$\mathbf{k}(\mathbf{u}) \triangleq \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}[(\dot{q}_1 + \dot{q}_2)^2 - \dot{q}_1^2] \\ b_{21}\dot{q}_1^2 + b_{22}[(\dot{q}_1 + \dot{q}_2)^2 - \dot{q}_1^2] \end{bmatrix} = \mathbf{B}\{\dot{\mathbf{q}}\}^2. \tag{1.37}$$

Equation (1.28) can then be written as follows:

$$\ddot{\mathbf{x}} = \mathbf{A}\tau + \mathbf{k}. \tag{1.38}$$

For a given dynamic state $\mathbf{u}$ of the manipulator, we define the state acceleration set, $S_{\mathbf{u}}$, as the image set of the actuator torque set T under the linear mapping (1.38):

$$S_{\mathbf{u}} = \{\ddot{\mathbf{x}}^P | \ddot{\mathbf{x}}^P = \mathbf{A}\tau + \mathbf{k}, \tau \in T\}. \tag{1.39}$$

$S_{\mathbf{u}}$ is therefore the set of all possible accelerations at any given dynamic state $\mathbf{u}$ of the manipulator. Since the dynamic state $\mathbf{u}$ of the manipulator essentially specifies the velocity $\dot{\mathbf{x}}^P$ of the point P in (1.11) in any configuration, we can also interpret the state acceleration set $S_{\mathbf{u}}$ (the set of available accelerations) as the acceleration capability of the manipulator when the manipulator is moving with the velocity $\dot{\mathbf{x}}^P$ in a given configuration $\mathbf{q}$.

3.5 Characterization of the acceleration sets

Once the acceleration sets defined in the previous section have determined, one would like to characterize them. In this section, we define two properties which are useful in characterizing acceleration sets.

Figure 4 shows an acceleration set S in the acceleration plane $\ddot{\mathbf{x}}$, and two circles C_1 and C_2. The circle C_1 of radius r_1 is the smallest circle centered at the origin which completely encloses S. Its radius r_1 therefore represents the maximum (magnitude of the) available acceleration in S. The circle C_2 of radius r_2 is the largest circle centered at the origin which lies within S. Its radius r_2 therefore represents the largest (magnitude of) acceleration available in all directions.

We define the following two properties of S:

- the maximum acceleration of S: $a_{\max}(S) = r_1$,
- the isotropic acceleration of S: $a_{\mathrm{iso}}(S) = r_2$.

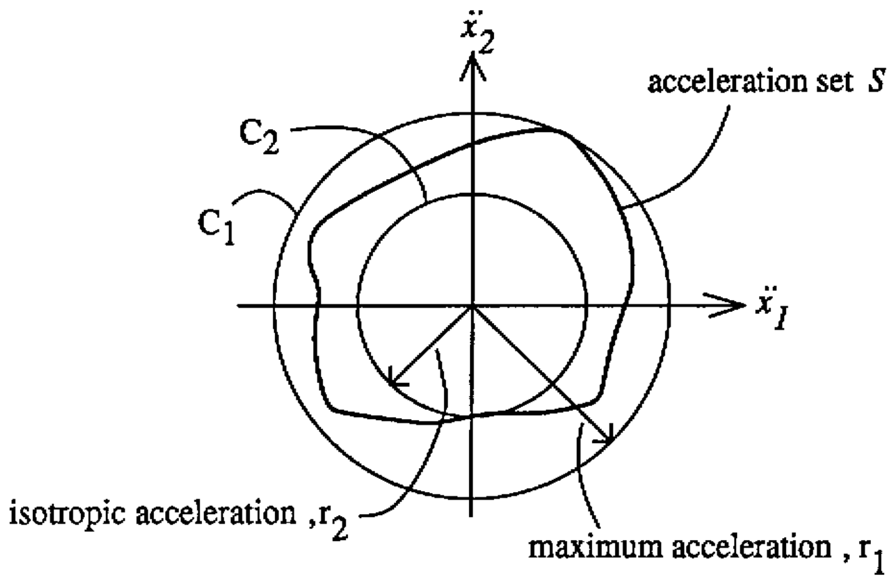

Figure 4: Characterization of an acceleration set in the acceleration plane

Comments:

1. As will be shown, the maximum acceleration and isotropic acceleration are two measures which can be readily extracted once the acceleration set is known.

2. The isotropic acceleration (Khatib and Burdick, 1987; Graettinger and Krogh, 1988) is a useful measure of the acceleration set, since it is a property which does not depend on direction.

3. The average acceleration of the set S cannot readily be extracted in closed-form (or by appropriate bounds) from the acceleration set S. It can however be numerically determined from descriptions of the various acceleration sets given in the next section. Also the physical meaning of the average acceleration is not clear.

4 Determination of the acceleration sets

Analytic expressions for the determination of the three sets S_τ, $S_{\dot{q}}$ and $S_{\mathbf{u}}$ are presented, respectively, in section 4.1, 4.2 and 4.3. The determination of S_τ and the state acceleration set $S_{\mathbf{u}}$ follows directly from well-known properties of linear mappings while the determination of the set $S_{\dot{q}}$ requires the derivation of the properties of quadratic mappings which are new

4.1 Determination of the image set S_τ

The set S_τ is the image set of the actuator torque set T under the linear mapping (1.32).

Result 1: The image set S_τ of the actuator torque set T under the linear mapping (1.32) is (the interior and boundary of) the parallelogram $A'B'C'D'$ in the $\ddot{\mathbf{x}}$ - plane whose vertices A', B', C', and D' are as follows:

$$\begin{aligned} A' &: (a_{11}\tau_{1o} + a_{21}\tau_{2o},\ a_{21}\tau_{1o} + a_{22}\tau_{2o}), \\ B' &: (a_{11}\tau_{1o} - a_{21}\tau_{2o},\ a_{21}\tau_{1o} - a_{22}\tau_{2o}), \\ C' &: (-a_{11}\tau_{1o} - a_{21}\tau_{2o},\ -a_{21}\tau_{1o} - a_{22}\tau_{2o}), \\ D' &: (-a_{11}\tau_{1o} + a_{21}\tau_{2o},\ -a_{21}\tau_{1o} + a_{22}\tau_{2o}), \end{aligned} \tag{1.40}$$

where a_{ij} (i,j=1,2) are the elements of the matrix **A** defined in equation (1.26). The centroid of the parallelogram $A'B'C'D'$ is the origin O of the $\ddot{\mathbf{x}}$-plane.

Result 2: The sides $A'B'$, $B'C'$, $C'D'$, and $D'A'$ of the parallelogram S_τ (Figure 5), which comprise the boundary of the set are given by the following equations:

$$A'B' : a_{22}\ddot{x}_1 - a_{12}\ddot{x}_2 = \det(A)\tau_{1o}, \tag{1.41}$$

$$B'C' : -a_{21}\ddot{x}_1 + a_{11}\ddot{x}_2 = \det(A)\tau_{2o}, \tag{1.42}$$

$$C'D' : a_{22}\ddot{x}_1 - a_{12}\ddot{x}_2 = -\det(A)\tau_{1o}, \tag{1.43}$$

$$D'A' : -a_{21}\ddot{x}_1 + a_{11}\ddot{x}_2 = -\det(A)\tau_{2o}. \tag{1.44}$$

where det(A) is the determinant of the (2×2) matrix **A**.

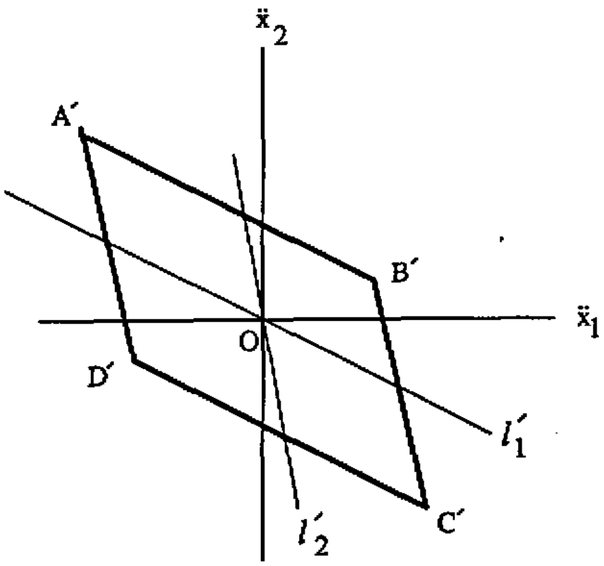

Figure 5: Image set of the linear mapping of a two degree-of-freedom planar manipulator

Proof of Result 1:

The following are well know properties of a linear mapping:

1. A line in the τ-plane will map into a line in the $\ddot{\mathbf{x}}$-plane. In particular, the line l_1, with equation $\tau_1 = 0$, maps into the line l_1' whose equation is

$$l_1' : a_{22}\ddot{x}_1 - a_{12}\ddot{x}_2 = 0, \tag{1.45}$$

and the line l_2, with equation $\tau_2 = 0$, maps into the line l_2' whose equation is

$$l_2' : -a_{21}\ddot{x}_1 + a_{11}\ddot{x}_2 = 0. \tag{1.46}$$

Both l_1' and l_2' pass through the origin (Figure 5).

2. Any line g_1 parallel to l_1 maps into a line g_1' parallel to l_1'.

3. Any line g_2 parallel to l_2 maps into a line g_2' parallel to l_2'.

Regarding the rectangle $ABCD$ (set T) as a set of lines parallel to l_1 and l_2 one can easily show the well-known fact that the image of $ABCD$ is a parallelogram $A'B'C'D'$. The vertices A', B', C' and D' are the images, respectively, of the vertices A, B, C and D. Substituting the coordinates of $A(\tau_{1o}, \tau_{2o})$, $B(\tau_{1o}, -\tau_{2o})$, $C(-\tau_{1o}, -\tau_{2o})$ and $D(-\tau_{1o}, \tau_{2o})$ into equation (1.32), we obtain the coordinates of the vertices A', B', C' and D' as given in equation (1.40). From equation (1.40), we see that the vertices A' and C' are equidistant from the origin and that the vertices B' and D' are equidistant from the origin. Therefore, the origin of the $\ddot{\mathbf{x}}$-plane is the centroid of the parallelogram $A'B'C'D'$.

Proof of Result 2:

We next need to determine the equations of the lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, which form the boundary of the parallelogram $A'B'C'D'$ in the $\ddot{\mathbf{x}}$ - plane. $A'B'$ is the image of the line AB. whose equation is $\tau = \tau_{1o}$; to obtain the equation of $A'B'$, substitute the equation of AB ($\tau = \tau_{1o}$) into (1.32) to obtain the following parametric equations in τ_2:

$$\ddot{x}_1 = a_{11}\tau_{1o} + a_{12}\tau_2, \qquad (1.47)$$

$$\ddot{x}_2 = a_{21}\tau_{1o} + a_{22}\tau_2. \qquad (1.48)$$

Eliminating the parameter τ_2 between (1.47) and (1.48), we obtain the equation of the line $A'B'$ in the $\ddot{\mathbf{x}}$ - plane as given by equation (1.41). In a similar fashion, we can obtain the equations of the lines $B'C'$, $C'D'$, and $D'A'$ as in equations (1.42) through (1.44). Note that from equations (1.41) through (1.44) we see that $A'B'$ is parallel to $C'D'$ and $B'C'$ is parallel to $D'A'$ so that $A'B'C'D'$ is indeed the parallelogram shown in Figure 5.

4.2 Determination of the image set $S_{\dot{\mathbf{q}}}$

The set $S_{\dot{\mathbf{q}}}$ is the image set of the joint variable rate set F under the mapping (1.33). Set $S_{\dot{\mathbf{q}}}$ is determined from the following results

Results:

The set F in the $\dot{\mathbf{q}}$ - plane is considered as a family of line segments passing through the origin. There are two such types of line segments: those which end on the boundaries J_1K_1 and J_2K_2 parallel to the

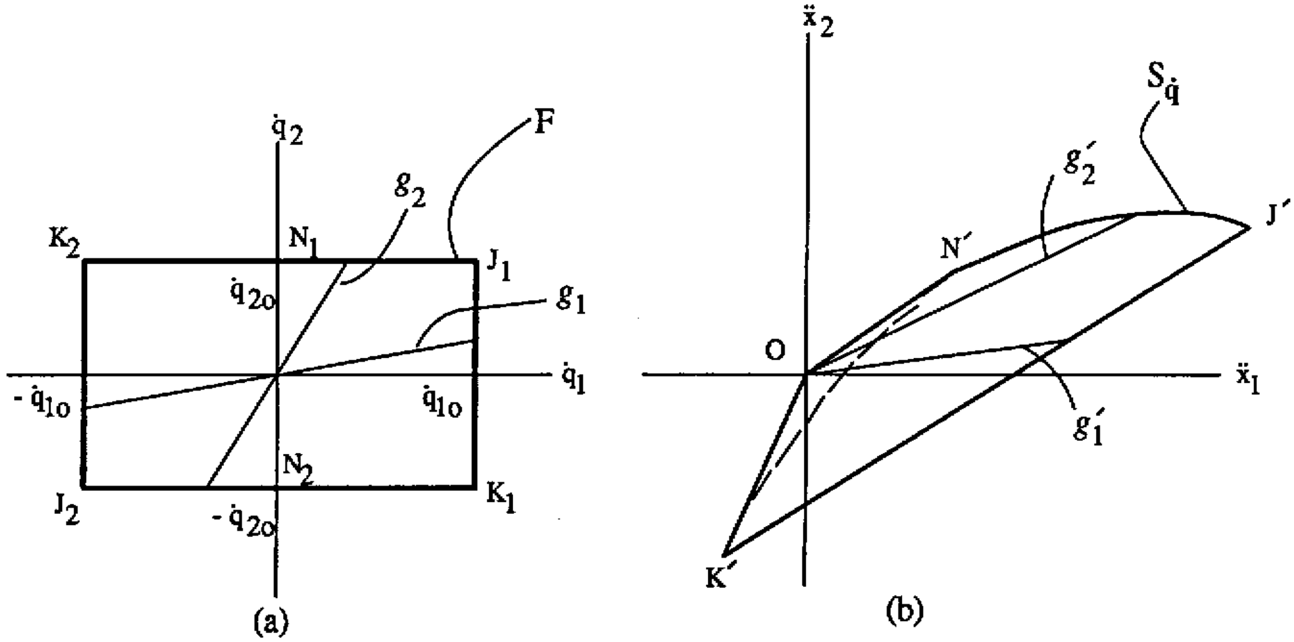

Figure 6: Image set of the quadratic mapping of a two degree-of-freedom manipulator

$\dot{q}_2$ - axis, a typical member of which is the line segment g_1 in Figure 6 (a), and those which end on the boundaries J_1K_2 and J_2K_1 parallel to the $\dot{q}_1$ - axis, a typical member of which is the line segment g_2 in Figure 6 (a).

1. Every line of the type g_1 maps into a line g_1' (see Figure 6 (b)) in the $\ddot{\mathbf{x}}$ - plane, one end of which is the origin and the other end of which lies on the line segment $J'K'$ whose equation is:

$$\frac{1}{b_{12}}\ddot{x}_1 - \frac{1}{b_{22}}\ddot{x}_2 + \left(-\frac{b_{11}}{b_{12}} + \frac{b_{21}}{b_{22}}\right)\dot{q}_{1o}^2 = 0 \tag{1.49}$$

   where $\ddot{x}_1$ lies in the interval $[b_{11}\dot{q}_{1o}^2 + b_{12}(\dot{q}_{2o}^2 + 2\dot{q}_{1o}\dot{q}_{2o}), b_{11}\dot{q}_{1o}^2 + b_{12}(\dot{q}_{2o}^2 - 2\dot{q}_{1o}\dot{q}_{2o})]$.

2. Every line of the type g_2 (see Figure 6 (b)) maps into a line g_2' in the $\ddot{x}$ plane, one end of which is the origin and the other end of which lies on the quadratic curve $K'N'J'$ (shown dashed between K' and N') whose equation in the parametric (in $\dot{q}_1$) form is:

$$\begin{aligned} \ddot{x}_1 &= b_{11}\dot{q}_1^2 + b_{12}(\dot{q}_{2o}^2 + 2\dot{q}_1\dot{q}_{2o}), \\ \ddot{x}_2 &= b_{21}\dot{q}_1^2 + b_{22}(\dot{q}_{2o}^2 + 2\dot{q}_1\dot{q}_{2o}). \end{aligned} \tag{1.50}$$

3. The image set of F is (the interior and boundary of) the region $ON'J'K'$ shown in Figure 6 (b), and the coordinates of N', J', and K' are as follows:

$$N' \quad (\; b_{12}\dot{q}_{2o}^2, \; b_{22}\dot{q}_{2o}^2), \tag{1.51}$$

$$J' \quad (\; b_{11} - b_{12})\dot{q}_{1o}^2 + b_{12}(\dot{q}_{1o} + \dot{q}_{2o})^2, \; (b_{21} - b_{22})\dot{q}_{1o}^2 + b_{22}(\dot{q}_{1o} + \dot{q}_{2o})^2), \tag{1.52}$$

$$K' \quad (\; (b_{11} - b_{12})\dot{q}_{1o}^2 + b_{12}(\dot{q}_{1o} - \dot{q}_{2o})^2, \; (b_{21} - b_{22})\dot{q}_{1o}^2 + b_{22}(\dot{q}_{1o} - \dot{q}_{2o})^2). \tag{1.53}$$

Proof:

The quadratic mapping is defined by the following equation (1.33):

$$\ddot{\mathbf{x}} = \mathbf{B}\{\dot{\mathbf{q}}\}^2$$

which can be written in the expanded form

$$\begin{aligned} \ddot{x}_1 &= b_{11}\dot{q}_1^2 + b_{12}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)^2, \\ \ddot{x}_2 &= b_{21}\dot{q}_1^2 + b_{22}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)^2. \end{aligned} \tag{1.54}$$

The determination of the set $S_{\dot{\mathbf{q}}}$ which is the image of the set F (Figure 6 (a)) consists of two steps:[2]

1. Establishment of the properties of the quadratic mapping, and

2. Determination of the boundary of the image set $S_{\dot{\mathbf{q}}}$.

Consider the (input) $\dot{\mathbf{q}}$ - plane. It is convenient to think of this plane as being generated by the continuous family of lines passing through the origin with parametric equation

$$\begin{cases} \dot{q}_1 = t \\ \dot{q}_2 = mt \end{cases}, \quad -\infty < m < \infty. \tag{1.55}$$

Each value of m gives us a member of the family of lines, a typical member of which is the line l shown in Figure 7. The image l' in the $\ddot{\mathbf{x}}$ - plane of the line l is obtained by substituting (1.55) into (1.54)

[2]While the approach described below is adequate for our present purposes, a more basic approach to determining $S_{\dot{\mathbf{q}}}$ is described in (Kim and Desa, 1989).

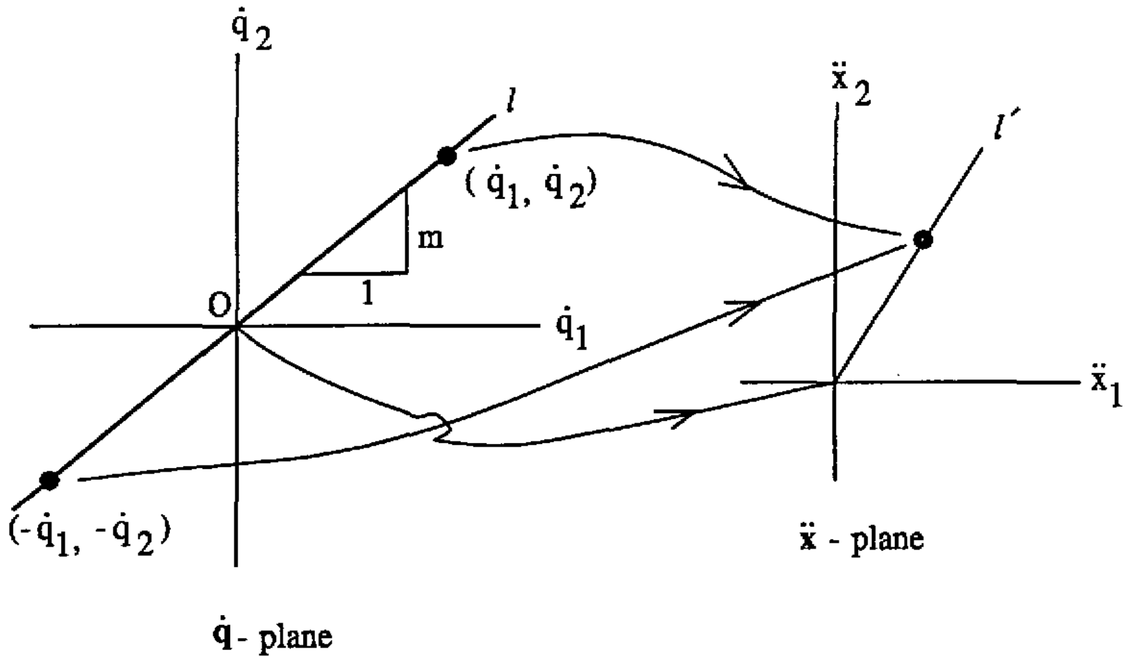

Figure 7: Properties of the quadratic mapping

and is described by the following parametric equation,

$$\begin{aligned} \ddot{x}_1 &= [b_{11} + b_{12}(m^2 + 2m)]t^2, \\ \ddot{x}_2 &= [b_{21} + b_{22}(m^2 + 2m)]t^2. \end{aligned} \tag{1.56}$$

From equation (1.55) and (1.56), one can enumerate the following facts:

Fact 1. The image of l, *viz.* l', is a straight line.

Fact 2. The origin of the $\dot{\mathbf{q}}$ - plane maps into the origin of the $\ddot{\mathbf{x}}$ - plane.

Fact 3. Two distinct points $(\dot{q}_1, \dot{q}_2)$ and $(-\dot{q}_1, -\dot{q}_2)$ map into the same point of the $\ddot{\mathbf{x}}$ - plane.

These results are shown graphically in Figure 7.

Fact 1 follows from the fact that (1.56) is the equation of a straight line in the parameter t. Fact 2 follows from the fact that the point (0, 0) in the $\dot{\mathbf{q}}$ - plane, represented by the parameter t = 0, maps into the point (0, 0) in the $\ddot{\mathbf{x}}$ - plane. If t is the parameter corresponding to the point $(\dot{q}_1, \dot{q}_2)$ in the

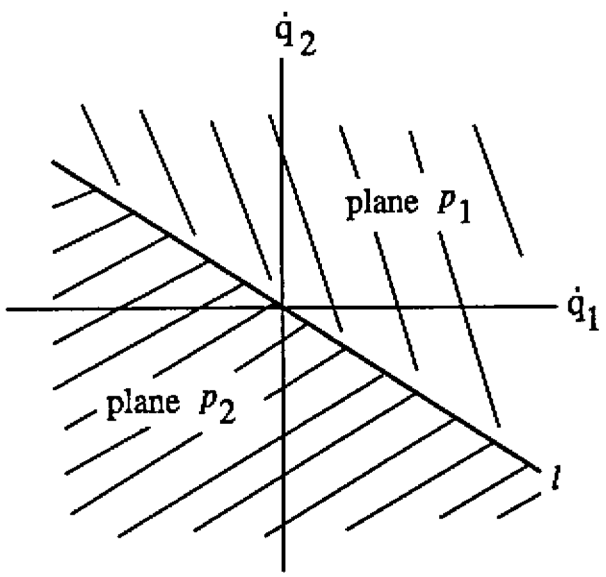

Figure 8: Input $\dot{\mathbf{q}}$ - plane

$\dot{\mathbf{q}}$-plane, then $-t$ is the parameter of the point $(-\dot{q}_1, -\dot{q}_2)$ from (1.55). From (1.56), we see that points with parameters t and $-t$ will map into the same point in the $\ddot{\mathbf{x}}$-plane. This proves Fact 3.

We can therefore state the following properties of the quadratic mapping:

Property 1. The image of any line segment in the $\dot{\mathbf{q}}$-plane, one end of which is the origin, is a line segment in the $\ddot{\mathbf{x}}$-plane with one end at the origin of the $\ddot{\mathbf{x}}$-plane.

Property 2. The image of the line l passing through the origin in the $\dot{\mathbf{q}}$ - plane is the half-line $\bar{l}$, one end of which is the origin (see Figure 7).

Property 3. Given any line l passing through the origin, and the two half-planes p_1 and p_2 formed by it (see Figure 8), p_1 and p_2 will have the same image set in the $\ddot{\mathbf{x}}$ - plane.

Property 1 is a direct consequence of Facts 1 and 2, property 2 a direct consequence of Facts 1, 2 and 3 and property 3 follows from Fact 3.

We now apply the above properties to determine the image set $S_{\dot{\mathbf{q}}}$ of the set F in the $\dot{\mathbf{q}}$-plane (See Figure 6). Property 3 tells us that if we "bisect" F into two "half-sets" with a line passing through the origin, then we only need to determine the image of one of these "half-sets". The most convenient half-set

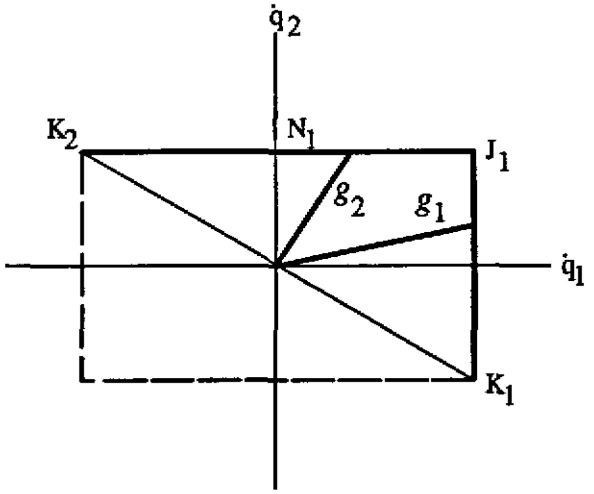

Figure 9: Input set F in $\dot{q}$ - plane

for our purposes is the set $K_1J_1K_2$ (see Figure 9), one of the two half-sets formed by the "bisecting" line K_1K_2.

In order to use Property 1, it is convenient to view the set $K_1J_1K_2$ as a family of line segments passing through the origin; we now need to determine the image of any line segment passing through the origin in this set. There are two cases to consider: the family of lines such as g_1, shown in Figure 9, which have one endpoint on the origin and the other endpoint on the line segment K_1J_1 and the family of lines such as g_2, also shown in Figure 9, which have one endpoint on the origin and the other endpoint on the line segment J_1K_2.

It is convenient to decompose the half-set $K_1J_1K_2$ into two subsets OK_1J_1 and OK_2J_1 as shown in Figure 10. Subset OK_1J_1 includes only the families of lines such as g_1 while subset OK_2J_1 includes only the families of lines such as g_2. The desired image set is the union of the images of OK_1J_1 and OK_2J_1. From property 1 of the quadratic mapping, we know that any line segment, such as g_1 or g_2 of the set F, will map into a line segment with one endpoint passing through the origin of the $\ddot{x}$ - plane. To obtain the other endpoint of the images of the two families of lines, we need to find the image of the line segment J_1K_1 of the subset OJ_1K_1 and the image of the line segment J_1K_2 of the subset OJ_1K_2.

First, we determine the image of subset OJ_1K_1 (10 (b)) by finding the image of K_1J_1. The equation

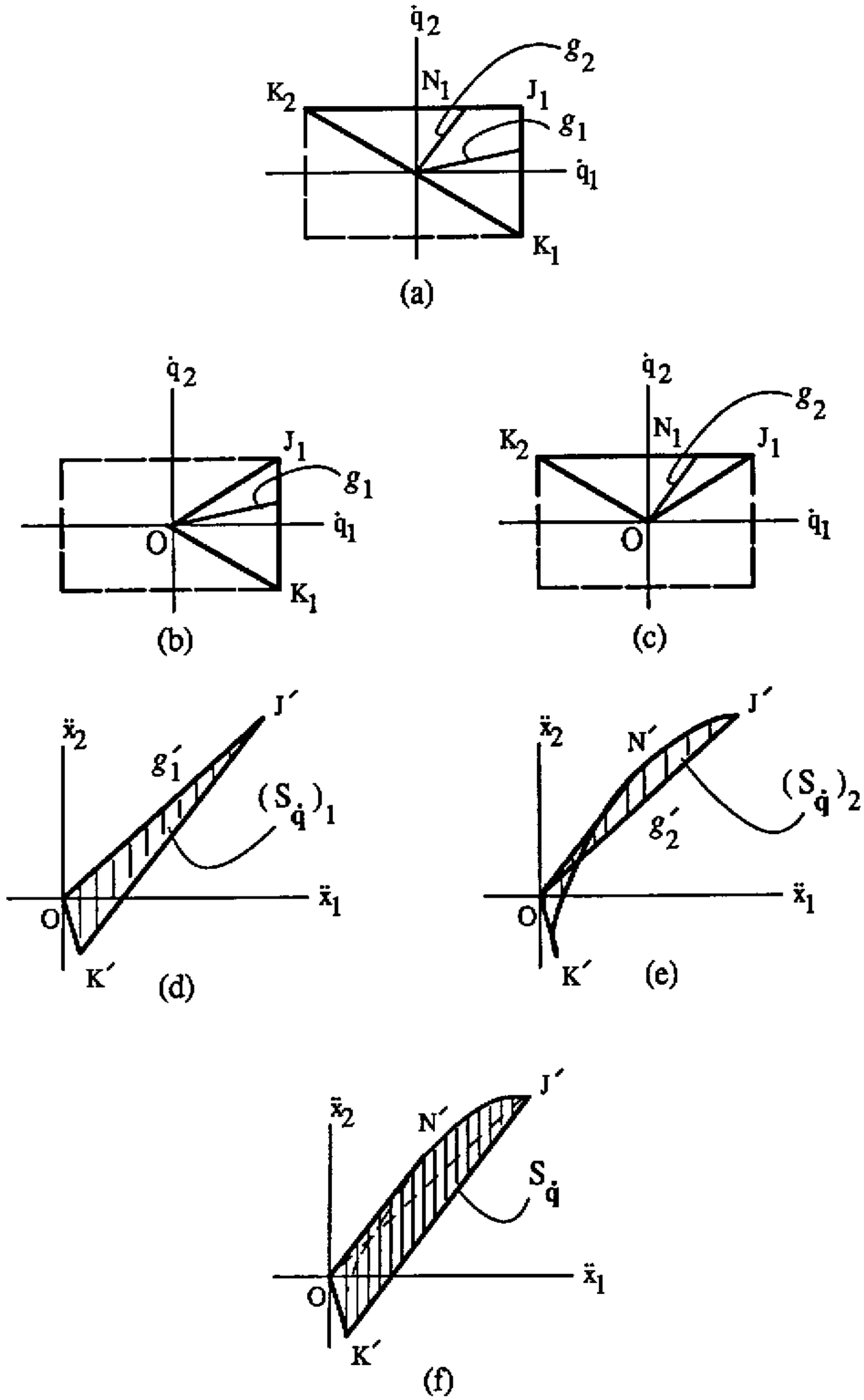

Figure 10: Determination of the image set $S_{\dot{q}}$

of K_1J_1 is

$$\dot{q}_1 = \dot{q}_{1o}; \ \mid \dot{q}_2 \mid \leq \dot{q}_{2o}. \tag{1.57}$$

To obtain the image $K'J'$ of K_1J_1, we substitute (1.57) into (1.54) to obtain

$$\begin{aligned} \ddot{x}_1 &= b_{11}\dot{q}_{1o}^2 + b_{12}(\dot{q}_2^2 + 2\dot{q}_{1o}\dot{q}_2)^2, \\ \ddot{x}_2 &= b_{21}\dot{q}_{1o}^2 + b_{22}(\dot{q}_2^2 + 2\dot{q}_{1o}\dot{q}_2)^2. \end{aligned} \tag{1.58}$$

Defining a parameter t as

$$t \triangleq \dot{q}_2^2 + 2\dot{q}_{1o}\dot{q}_2, \tag{1.59}$$

equation (1.58) can be written in the form

$$\ddot{x}_1 = b_{11}\dot{q}_{1o}^2 + b_{12}t, \tag{1.60}$$

$$\ddot{x}_2 = b_{21}\dot{q}_{1o}^2 + b_{22}t. \tag{1.61}$$

Eliminating the parameter t between the two equations, we obtain the equation for the line segment $J'K'$ as given by equation (1.49). (This proves the first part of the result.) The resulting subset $(S_{\dot{q}})_1$ which is the image of OK_1J_1 is shown in Figure 10 (d).

Next, we determine the image of subset OJ_1K_2 by finding the images of K_2J_1. To obtain the image $J'N'K'$ of J_1K_2, we substitute the equation for J_1K_2,

$$\dot{q}_2 = \dot{q}_{2o}; \ \mid \dot{q}_1 \mid \leq \dot{q}_{1o} \tag{1.62}$$

into (1.54) to obtain the parametric equation (1.50). Note that (1.50) represents the equation of a quadratic curve in terms of the parameter $\dot{q}_1$. (This completes the second part of the result.) The resulting subset $(S_{\dot{q}})_2$ which is the image of OK_2J_1 is shown in Figure 10 (e).

The desired image set $S_{\dot{q}}$ of F is the union of $(S_{\dot{q}})_1$ and $(S_{\dot{q}})_2$ and is shown in Figure 10 (f).

Note that the intersection of $(S_{\dot{q}})_1$ and $(S_{\dot{q}})_2$, $(S_{\dot{q}})_1 \cap (S_{\dot{q}})_2$, is not empty. Because $S_{\dot{q}}$ is the image of F under a quadratic mapping, there are points inside $S_{\dot{q}}$ which are the image of two distinct points in $K_1J_1K_2$. In particular, any point in the set $(S_{\dot{q}})_1 \cap (S_{\dot{q}})_2$, will be the image of two points are belonging to OK_1J_1 and the other to OK_2J_1.

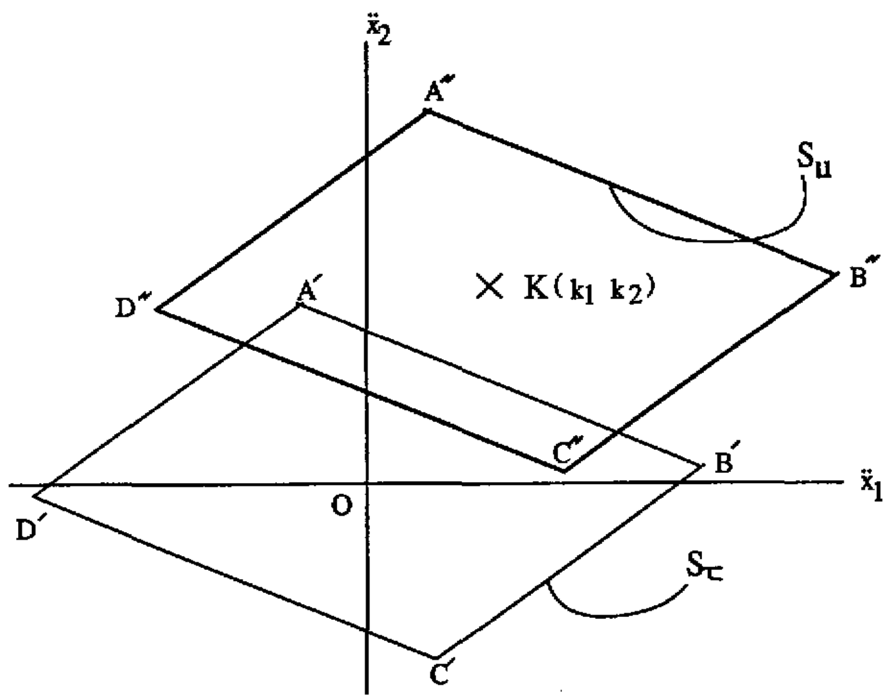

Figure 11: State acceleration set of a two degree-of-freedom manipulator

The images N', J' and K', respectively, of $N_1(0,\ \dot{q}_{2o})$, $J_1(\dot{q}_{1o},\ \dot{q}_{2o})$ and $K_1(\dot{q}_{1o},\ -\dot{q}_{2o})$, are obtained by substituting their $(\dot{q}_1, \dot{q}_2)$ coordinates into equation (1.54) to obtain the required results (1.51), (1.52), and (1.53).

4.3 Determination of the state acceleration set $S_\mathbf{u}$

The state acceleration $S_\mathbf{u}$ corresponding to a state $\mathbf{u} = (\mathbf{q}, \dot{\mathbf{q}})^T$ of the planar manipulator was defined by equation (1.39) and is the image set of the actuator torque set T under the mapping (1.38). We obtain the following results for the determination of the state acceleration set $S_\mathbf{u}$.

Result 1: For every element $\ddot{\mathbf{x}}(S_\tau)$ of the image set S_τ, there is a corresponding element $\ddot{\mathbf{x}}(S_\mathbf{u})$ of the state acceleration set $S_\mathbf{u}$, given by

$$\ddot{\mathbf{x}}(S_\mathbf{u}) = \ddot{\mathbf{x}}(S_\tau) + \mathbf{k}(\mathbf{q},\ \dot{\mathbf{q}}), \tag{1.63}$$

where

$$\mathbf{k}(\mathbf{q},\ \dot{\mathbf{q}}) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} b_{11}\dot{q}_1^2 + b_{12}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) \\ b_{21}\dot{q}_1^2 + b_{22}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) \end{bmatrix} = \mathbf{B}\{\dot{\mathbf{q}}\}^2. \tag{1.64}$$

Result 2: The state acceleration set $S_{\mathbf{u}}$, corresponding to a state $\mathbf{u} = (\mathbf{q},\ \dot{\mathbf{q}})^T$ of the planar two degree-of-freedom manipulator is the parallelogram $A''B''C''D''$ shown in Figure 11 obtained by translating the set S_τ by the vector $\mathbf{k}(\mathbf{q},\ \dot{\mathbf{q}})$ in the $\ddot{\mathbf{x}}$ - plane. The centroid of $S_{\mathbf{u}}$ is (k_1, k_2).

Proof of Result 1:

The results **1** and **2** are straightforward.

From (1.34), a member $\ddot{\mathbf{x}}(S_\tau)$ of S_τ is given by

$$\ddot{\mathbf{x}}(S_\tau) = \mathbf{A}\tau. \tag{1.65}$$

From (1.39), a member $\ddot{\mathbf{x}}(S_{\mathbf{u}})$ of $S_{\mathbf{u}}$ is given by

$$\ddot{\mathbf{x}}(S_{\mathbf{u}}) = \mathbf{A}\tau + \mathbf{k} \tag{1.66}$$

where $\mathbf{k}$ is given by equation (1.64). Combining (1.65) and (1.66), we obtain

$$\ddot{\mathbf{x}}(S_{\mathbf{u}}) = \ddot{\mathbf{x}}(S_\tau) + \mathbf{k}, \tag{1.67}$$

which is equation (1.63).

Proof of result 2:

From equation (1.63), we see that if we take a vector $\ddot{\mathbf{x}}(S_\tau)$ of S_τ and add the vector $\mathbf{k}$ to it we obtain the corresponding member $\ddot{\mathbf{x}}(S_{\mathbf{u}})$ of $S_{\mathbf{u}}$. So, if we add the vector $\mathbf{k}$ to every vector in the set S_τ we obtain the required set $S_{\mathbf{u}}$. Therefore, $S_{\mathbf{u}}$ is the parallelogram $A''B''C''D''$ (Figure 11) obtained by translating the set S_τ (the parallelogram $A'B'C'D'$ in Figure 11) by the vector $\mathbf{k}$. The centroid of S_τ is $\ddot{\mathbf{x}}(S_\tau) = (0, 0)$. From (1.67), we see that the corresponding centroid of $S_{\mathbf{u}}$ is

$$\ddot{\mathbf{x}}(S_{\mathbf{u}}) = 0 + \mathbf{k} = \mathbf{k}. \tag{1.68}$$

This completes the proof of Result 2.

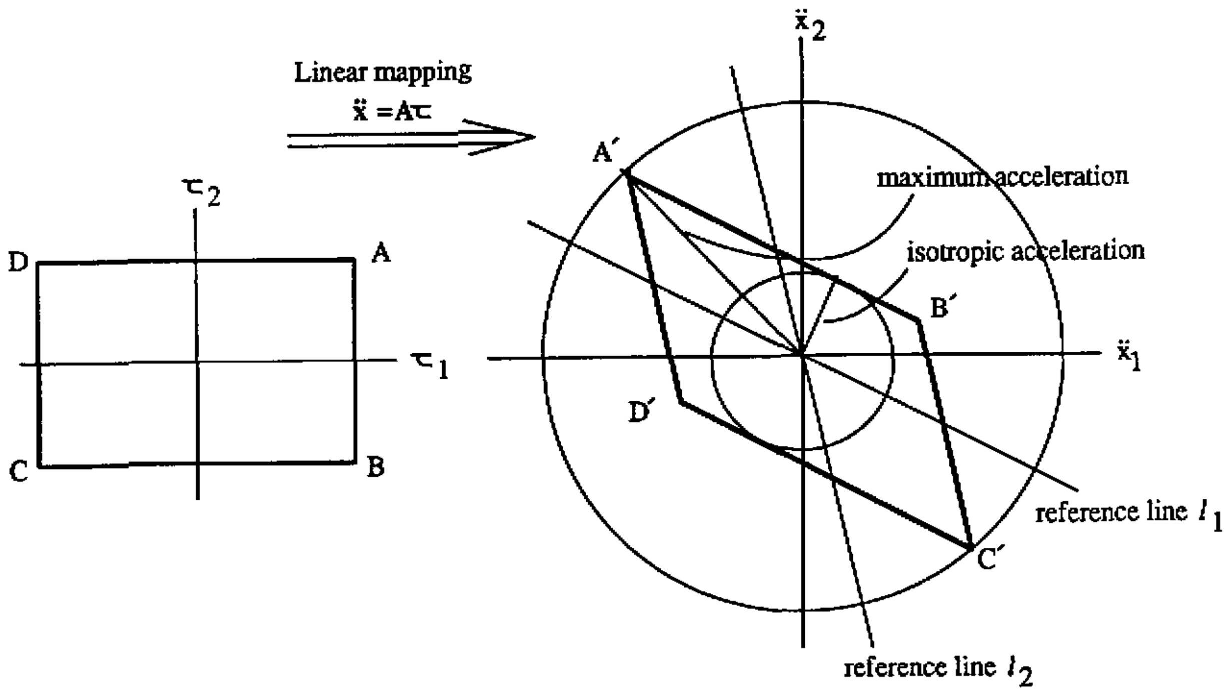

Figure 12: Characterization of the image set of the linear mapping for a two degree-of-freedom manipulator

5 Properties of the acceleration sets

In this section, we extract the properties - defined in subsection 3.5 - of the acceleration sets S_τ, $S_{\dot{\mathbf{q}}}$ and $S_{\mathbf{u}}$ determined in the previous section.

5.1 Properties of the acceleration set S_τ

We characterize the image set S_τ of the linear mapping as follows:

Result 1: The maximum acceleration of the acceleration set S_τ is denoted by $a_{\max}(S_\tau)$ and is given by

$$a_{\max}(S_\tau) = \max[d(OA'),\ d(OB')] \tag{1.69}$$

where

$$\begin{aligned} d(OA') &= \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o})^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o})^2} \\ d(OB') &= \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o})^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o})^2} \end{aligned}$$

Result 2: The isotropic acceleration of the acceleration set S_τ will be denoted by $a_{iso}(S_\tau)$ and is given by

$$a_{iso}(S_\tau) = \min[\rho(A'B'),\ \rho(B'C')] \tag{1.70}$$

where

$$\rho(A'B') = \frac{|\det A|\ \tau_{1o}}{\sqrt{a_{12}^2 + a_{22}^2}},$$
$$\rho(B'C') = \frac{|\det A|\ \tau_{2o}}{\sqrt{a_{11}^2 + a_{21}^2}}.$$

Proof of Result 1:

The maximum acceleration of S_τ is the distance from the origin to the furthest vertex of the parallelogram $A'B'C'D'$ (see Figure 12). Letting $d\,(OA')$ through $d\,(OD')$ denote, respectively, the distances of vertices A' through D' from the origin O in the $\ddot{x}$ - plane, $a_{max}(S_\tau)$ is given by

$$a_{max}(S_\tau) = \max[\ d(OA'),\ d(OB'),\ d(OC'),\ d(OD')\ ]. \tag{1.71}$$

A' and C' are equidistant from the origin O. Also, B' and D' are equidistant from the origin O. So, $a_{max}(S_\tau)$ is given by

$$a_{max}(S_\tau) = \max[\ d(OA'),\ d(OB')\ ]. \tag{1.72}$$

Using (1.32), the distance $d(OA')$ from the origin O to the point A'

$$d(OA') = \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o})^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o})^2}. \tag{1.73}$$

In exactly analogous fashion, we obtain

$$d(OB') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o})^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o})^2}. \tag{1.74}$$

Equations (1.72), (1.73) and (1.74) comprise Result 1.

Proof of Result 2:

The isotropic acceleration of S_τ is the shortest distance from the origin to the sides of the parallelogram $A'B'C'D'$. Letting $\rho\ (A'B')$, $\rho\ (B'C')$, $\rho\ (C'D')$, and $\rho\ (D'A')$ denote, respectively, the perpendicular distances from O to the sides $A'B'$, $B'C'$, $C'D'$, and $D'A'$, $a_{iso}(S_\tau)$ is given by

$$a_{iso}(S_\tau) = \min[\rho(A'B'),\ \rho(B'C'),\ \rho(C'D'),\ \rho(D'A')]. \tag{1.75}$$

Since the origin O is the centroid of the parallelogram S_τ, parallel lines of the parallelogram $A'B'C'D'$ must be equidistant from the origin. Therefore, we can write the following relations:

$$\rho(A'B') = \rho(C'D'), \tag{1.76}$$

$$\rho(B'C') = \rho(D'A'). \tag{1.77}$$

Using (1.76) and (1.77), (1.75) can be written

$$a_{iso}(S_\tau) = \min[\rho(A'B'),\ \rho(B'C')]. \tag{1.78}$$

The distance from a point $P(x_o,\ y_o)$ to line $ax + by + k = 0$ is given by the following well-known result:

$$\frac{|\ ax_o + by_o + k\ |}{\sqrt{a^2 + b^2}}. \tag{1.79}$$

Using equation (1.41), (1.42) and (1.79), we obtain

$$\rho(A'B') = \frac{|\ \det \mathbf{A}\ |\ \tau_{1o}}{\sqrt{a_{12}^2 + a_{22}^2}}, \tag{1.80}$$

$$\rho(B'C') = \frac{|\ \det \mathbf{A}\ |\ \tau_{2o}}{\sqrt{a_{11}^2 + a_{21}^2}}. \tag{1.81}$$

Substituting (1.80) and (1.81) into equation (1.78), we can obtain the required result (1.70) for the isotropic acceleration $a_{iso}(S_\tau)$.

5.2 Properties of the acceleration set $S_{\dot{\mathbf{q}}}$

We characterize the image set $S_{\dot{\mathbf{q}}}$ by the maximum acceleration and the maximum distance of any element of $S_{\dot{\mathbf{q}}}$ from the two references lines l_1 and l_2 shown in Figure 5.

Definition 1:

$$\begin{aligned} l(\dot{\mathbf{q}}) &\triangleq l(\dot{q}_1,\ \dot{q}_2) \\ &\triangleq \sqrt{(b_{11}\dot{q}_1^2 + b_{12}\dot{q}_2^2 + 2b_{12}\dot{q}_1\dot{q}_2)^2 + (b_{21}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + 2b_{22}\dot{q}_1\dot{q}_2)^2} \end{aligned} \tag{1.82}$$

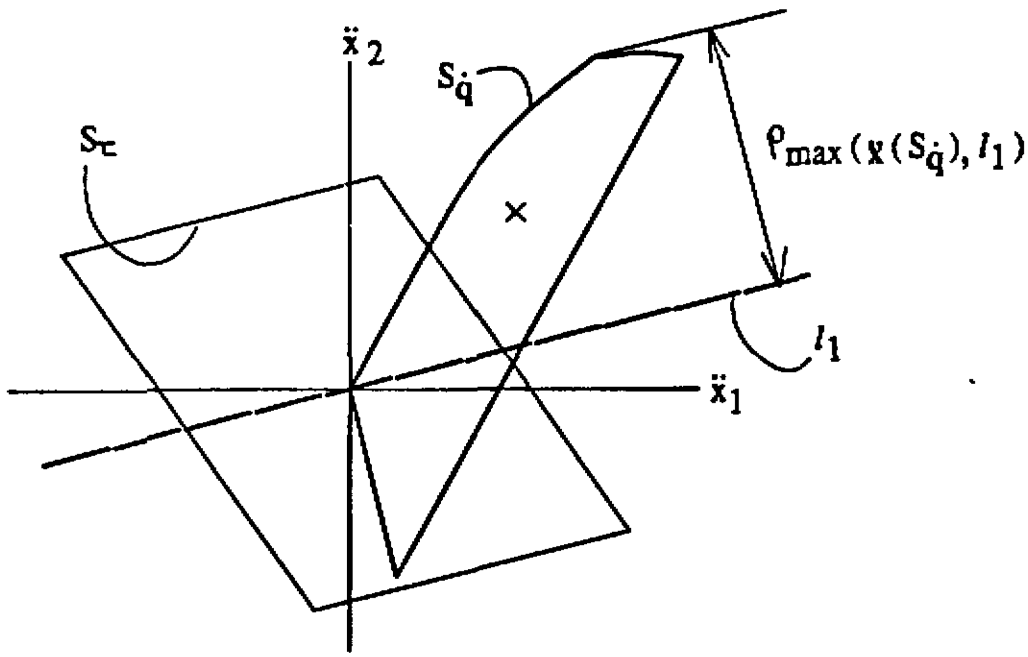

Figure 13: Maximum distance from reference line l_1 to a point on the set $S_{\dot{q}}$

Definition 2: Let $\dot{q}'$ denote the real solution of the following cubic equation in $\dot{q}_1$:

$$\begin{aligned} & [b_{11}\dot{q}_1^2 + b_{12}(\dot{q}_{2o}^2 + 2\dot{q}_1\dot{q}_{2o})](b_{11}\dot{q}_1 + b_{12}\dot{q}_2) \\ + \quad & [b_{21}\dot{q}_1^2 + b_{22}(\dot{q}_{2o}^2 + 2\dot{q}_1\dot{q}_{2o})](b_{21}\dot{q}_1 + b_{22}\dot{q}_2) = 0. \end{aligned} \tag{1.83}$$

Definition 3: Let $\rho(\ddot{x}(S_{\dot{q}}), l_1)$ and $\rho(\ddot{x}(S_{\dot{q}}), l_2)$ denote, respectively, the distance of any point $\ddot{x}(S_{\dot{q}})$ of $S_{\dot{q}}$ from the lines l_1 and l_2.

$$\rho_{max}(\ddot{x}(S_{\dot{q}}), l_1) \triangleq \max \rho(\ddot{x}(S_{\dot{q}}), l_1), \tag{1.84}$$

$$\rho_{max}(\ddot{x}(S_{\dot{q}}), l_2) \triangleq \max \rho(\ddot{x}(S_{\dot{q}}), l_2). \tag{1.85}$$

$\rho_{max}(\ddot{x}(S_{\dot{q}}), l_1)$, for example, represents the distance of that point of $S_{\dot{q}}$ furthest from l_1; $\rho_{max}(\ddot{x}(S_{\dot{q}}), l_1)$ and $\rho_{max}(\ddot{x}(S_{\dot{q}}), l_2)$ are necessary for determining the local isotropic acceleration in subsubsection 6.

Definition 4:

$$\dot{q}'' \triangleq \frac{a_{22}b_{11} - a_{12}b_{21}}{a_{22}b_{12} - a_{12}b_{22}}\,\dot{q}_{2o}. \tag{1.86}$$

Definition 5:

$$\sigma_1(\dot q_1,\ \dot q_2) \triangleq \frac{|\,a_{22}\ddot x_1(\dot q_1,\ \dot q_2) - a_{12}\ddot x_2(\dot q_1,\ \dot q_2)\,|}{\sqrt{a_{22}^2 + a_{12}^2}}$$
$$= \frac{|\,a_{22}[b_{11}\dot q_1^2 + b_{12}(\dot q_2^2 + 2\dot q_1\dot q_2)] - a_{12}[b_{21}\dot q_1^2 + b_{22}(\dot q_2^2 + 2\dot q_1\dot q_2)]\,|}{\sqrt{a_{21}^2 + a_{12}^2}} \qquad (1.87)$$

$$\sigma_2(\dot q_1,\ \dot q_2) \triangleq \frac{|\,a_{21}\ddot x_1(\dot q_1,\ \dot q_2) - a_{11}\ddot x_2(\dot q_1,\ \dot q_2)\,|}{\sqrt{a_{22}^2 + a_{11}^2}}$$
$$= \frac{|\,a_{21}[b_{11}\dot q_1^2 + b_{12}(\dot q_2^2 + 2\dot q_1\dot q_2)] - a_{11}[b_{21}\dot q_1^2 + b_{22}(\dot q_2^2 + 2\dot q_1\dot q_2)]\,|}{\sqrt{a_{21}^2 + a_{11}^2}} \qquad (1.88)$$

Result 1: For a general two degree-of-freedom planar manipulator, the maximum acceleration of the acceleration set $S_{\dot q}$ will be denoted by $a_{max}(S_{\dot q})$ and is given by

$$a_{max}(S_{\dot q}) = \max[l(\dot q_{1o},\ -\dot q_{1o}),\ l(\dot q_1',\ \dot q_{2o}),\ l(\dot q_{1o},\ \dot q_{2o}),\ l(\dot q_{1o},\ -\dot q_{2o})] \qquad (1.89)$$

where $\dot q_1'$ is defined in (1.83).

For the two degree-of-freedom open-loop planar manipulator, shown in Figure 1,

$$a_{max}(S_{\dot q}) = l(\dot q_{1o}, \dot q_{2o}). \qquad (1.90)$$

Result 2: For a general two degree-of-freedom manipulator, the maximum distance from an element of $S_{\dot q}$ to the reference lines l_1 and l_2 are, respectively, given by

$$\rho_{max}\ \ (\ \ \ddot x(S_{\dot q}), l_i)] \qquad (1.91)$$
$$= \max[\sigma_i(\dot q_{1o},\ -\dot q_{1o}),\ \sigma_i(\dot q'',\ \dot q_{2o}),\ \sigma_i(\dot q_{1o},\ \dot q_{2o}),\ \sigma_i(\dot q_{1o},\ -\dot q_{2o})],\ i = 1,2. \qquad (1.92)$$

where $\dot q''$ is defined in (1.86).

Proof of Result 1:

The magnitude squared of the acceleration of a point $\ddot x(S_{\dot q})$ of $S_{\dot q}$, denoted by $a^2(S_{\dot q})$, is given by

$$a^2(S_{\dot q}) \triangleq l^2(\dot q_1,\ \dot q_2) = \ddot x_1^2(\dot q_1,\ \dot q_2) + \ddot x_2^2(\dot q_1,\ \dot q_2)$$
$$= (b_{11}\dot q_1^2 + b_{12}\dot q_2^2 + 2b_{12}\dot q_1\dot q_2)^2 + (b_{21}\dot q_1^2 + b_{22}\dot q_2^2 + 2b_{22}\dot q_1\dot q_2)^2. \qquad (1.93)$$

The maximum magnitude squared of the acceleration for the set $S_{\dot{\mathbf{q}}}$, denoted by $a^2_{\max}(S_{\dot{\mathbf{q}}})$, is given by

$$a^2_{\max}(S_{\dot{\mathbf{q}}}) \;=\; \max_{(\dot{\mathbf{q}}\in F)} l^2(\dot{q}_1,\ \dot{q}_2), \tag{1.94}$$

where F is shown in Figure 2 and specified by the constraints

$$|\ \dot{q}_1\ | \leq \dot{q}_{1o}, \tag{1.95}$$

$$|\ \dot{q}_2\ | \leq \dot{q}_{2o}. \tag{1.96}$$

The maximum of l^2, required in equation (1.94), will occur at point $\dot{\mathbf{q}} \in F$ which is either inside F or on the boundaries of F where one or both constraints might be active. Furthermore, because of property 3 of the quadratic mapping, we only need to look at the boundaries J_1K_1 and J_1K_2 of the half-set $K_2J_1K_1$. Therefore, to obtain the maximum of (1.82) under the constraints (1.95) and (1.96), we should consider the following possibilities:

1. Neither of the constraints is active, i.e., the $\max[l^2(\dot{q}_1,\ \dot{q}_2)]$ occurs at a point $\dot{\mathbf{q}}$ inside F.

2. Constraint (1.95) is active, i.e., $\max[l^2(\dot{q}_1,\ \dot{q}_2)]$ occurs at a point $\dot{\mathbf{q}}$ lying on the boundary J_1K_1 of F.

3. Constraint (1.96) is active, i.e., $\max[l^2(\dot{q}_1,\ \dot{q}_2)]$ occurs at a point $\dot{\mathbf{q}}$ lying on the boundary J_1K_2 of F.

4. Both constraints are active, i.e., $\max[l^2(\dot{q}_1,\ \dot{q}_2)]$ occurs at either (a) point J_1 (b) point K_1 or (c) point K_2. Since, by virtue of Fact 3 of subsubsection 3.1.2, points K_1 and K_2 have the same image, we only need to consider either K_1 or K_2: we will choose K_1.

To obtain the conditions for each one of the above cases to yield a maximum, we first differentiate $l^2(\dot{q}_1, \dot{q}_2)$ with respect to $\dot{q}_1$ and $\dot{q}_2$ to obtain

$$\begin{aligned}\frac{\partial(l^2)}{\partial \dot{q}_1} &= 4[b_{11}\dot{q}_1^2 + b_{12}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)](b_{11}\dot{q}_1 + b_{12}\dot{q}_2)\\ &+ 4[b_{21}\dot{q}_1^2 + b_{22}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)](b_{21}\dot{q}_1 + b_{22}\dot{q}_2),\end{aligned} \tag{1.97}$$

$$\begin{aligned}\frac{\partial(l^2)}{\partial \dot{q}_2} &= 4[b_{11}\dot{q}_1^2 + b_{12}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)]b_{12}(\dot{q}_1 + \dot{q}_2)\\ &+ 4[b_{21}\dot{q}_1^2 + b_{22}(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)]b_{22})(\dot{q}_1 + \dot{q}_2).\end{aligned} \tag{1.98}$$

Now, we consider each case.

Case 1

To obtain the required $\dot{\mathbf{q}}$ for the case where both constraints are inactive, we set the right-hand side of (1.97) and (1.98) to zero,

$$\frac{\partial l^2}{\partial \dot{q}_1} = 0 \text{ and } \frac{\partial l^2}{\partial \dot{q}_2} = 0 \tag{1.99}$$

and obtain

$$\dot{q}_1 = \dot{q}_2 = 0 \tag{1.100}$$

which actually corresponds to the minimum value of $l^2(\dot{q}_1,\ \dot{q}_2)$, *viz*, zero. Therefore, $\max(l^2)$ does not occur at a point $\dot{\mathbf{q}}$ inside F.

Case 2

Since constraint (1.95) is active on the boundary J_1K_1 of F, we have

$$\dot{q}_1 = \dot{q}_{1o} \text{ (constant).} \tag{1.101}$$

To obtain the maximum of l^2, we set $\partial l^2/\partial \dot{q}_2 = 0$. We therefore set the right-hand side of (1.98) to zero to obtain

$$\dot{q}_1 + \dot{q}_2 = 0. \tag{1.102}$$

Combining (1.101) and (1.102), we obtain

$$\dot{q}_2 = -\dot{q}_{1o} \tag{1.103}$$

and

$$\max[l^2(\dot{q}_1,\ \dot{q}_2)] = l^2(\dot{q}_{1o},\ -\dot{q}_{1o}). \tag{1.104}$$

Case 3

Since constraint (1.96) is active on the boundary J_1K_2 of F, we have

$$\dot{q}_2 = \dot{q}_{2o} \text{ (constant).} \tag{1.105}$$

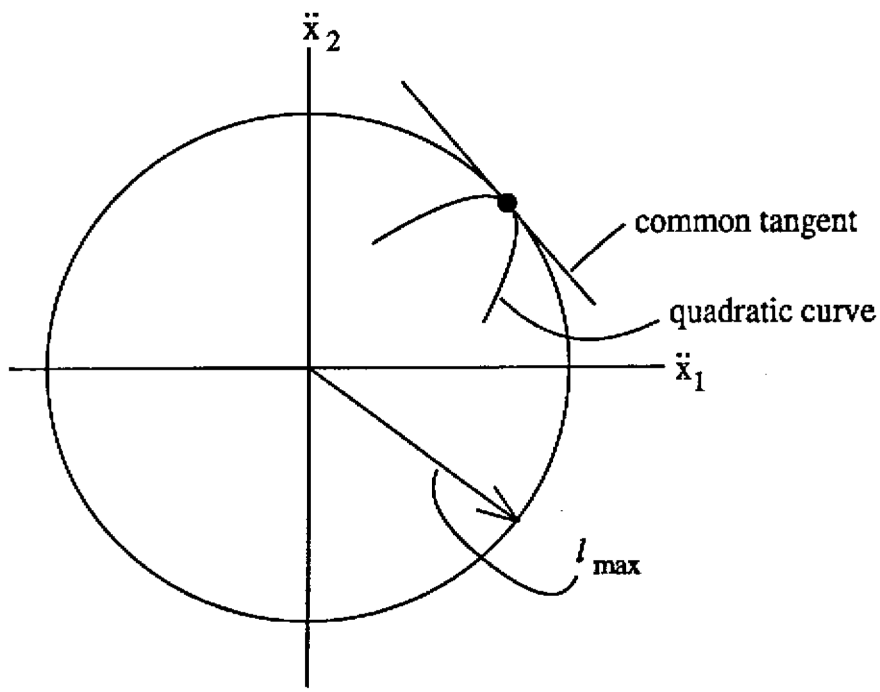

Figure 14: Common tangency between the quadratic curve and circle

To obtain the maximum, we set $\partial l^2/\partial \dot{q}_1 = 0$. We therefore set the right-hand side of (1.98) to zero and set $\dot{q}_2 = \dot{q}_{2o}$ to obtain

$$\begin{aligned} & [b_{11}\dot{q}_1^2 + b_{12}(\dot{q}_{2o}^2 + 2\dot{q}_1\dot{q}_{2o})](b_{11}\dot{q}_1 + b_{12}\dot{q}_{2o}) \\ + \quad & [b_{21}\dot{q}_1^2 + b_{22}(\dot{q}_{2o}^2 + 2\dot{q}_{1o}\dot{q}_2)](b_{21}\dot{q}_1 + b_{22}\dot{q}_{2o}) = 0. \end{aligned} \tag{1.106}$$

Equation (1.106) is a cubic in $\dot{q}_1$ and will therefore have three solutions. Using simple ideas from algebraic geometry, we now show that (1.106) can have at most one real solution in the region $|\dot{q}_1| < \dot{q}_{1o}$.

If $l^2(\dot{q}_1, \dot{q}_2)$ does have a maximum l_{max} then the condition $\partial l^2/\partial \dot{q}_1 = 0$ for obtaining l_{max} is the condition for the quadratic curve (1.50) -the image of J_1K_2 in the $\ddot{x}$-plane- and a circle of radius l_{max} to have a common tangent (see Figure 14). By Bezout's theorem (Semple and Roth, 1949), a quadratic curve and a circle can have at most two common points of tangency. Therefore, equation (1.106), which expresses the condition $\partial l^2/\partial \dot{q}_1 = 0$, can have at most two real roots (one for each point of tangency). However, (1.106) is a cubic in $\dot{q}_1$ and can therefore have either one real root or three real roots. Combining the last two statements, we see that (1.106) can have at most one real root. (Since we are looking at the quadratic curve in the region $|\dot{q}_1| < \dot{q}_{1o}$, the real solution of (1.106) might lie outside the constraints which simply means that (1.93) does not have an extremum in the region $|\dot{q}_1| < \dot{q}_{1o}$). Denoting the real solution (1.106) in the region $|\dot{q}_1| < \dot{q}_{1o}$ by $\dot{q}_1'$ and using (1.105), we can write

$$\max[l^2(\dot{q}_1, \dot{q}_2)] = l^2(\dot{q}_1', \dot{q}_{2o}). \tag{1.107}$$

Case 4-a

When both constraints (1.95) and (1.96) are active, and $\max[l^2(\dot{q}_1, \dot{q}_2)]$ occurs at $J_1(\dot{q}_{1o}, \dot{q}_{2o})$, then

$$\max[l^2(\dot{q}_1, \dot{q}_2)] = l^2(\dot{q}_{1o}, \dot{q}_{2o}). \tag{1.108}$$

Case 4-b

When both constraints (1.95) and (1.96) are active, and $\max[l^2(\dot{q}_1, \dot{q}_2)]$ occurs at $K_1(\dot{q}_{1o}, -\dot{q}_{2o})$, then

$$\max[l^2(\dot{q}_1, \dot{q}_2)] = l^2(\dot{q}_{1o}, -\dot{q}_{2o}). \tag{1.109}$$

Therefore, $a_{max}(S_{\dot{q}})$ is obtained as the maximum of four quantities defined by equations (1.104), (1.107), (1.108), and (1.109). This concludes the Proof of Result 1.

Proof of Result 2:

The distance of any point $\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}})$ of $S_{\dot{\mathbf{q}}}$ from the line l_i, i=1, 2, is given by

$$\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1) = \sigma_1(\dot{q}_1, \ \dot{q}_2) \tag{1.110}$$

$$\rho(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_2) = \sigma_2(\dot{q}_1, \ \dot{q}_2) \tag{1.111}$$

where σ_1 and σ_2 are defined in equations (1.87) and (1.88). We first wish to determine $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1)$ the distance of l_1 from that point of $S_{\dot{\mathbf{q}}}$ furthest away from it (l_1). $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1)$ is shown in Figure 13 for given $S_{\dot{\mathbf{q}}}$ and given reference line l_1 and can be defined as follows:

$$\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1) = \max_{(\dot{\mathbf{q}} \in F)} |\sigma_1(\dot{q}_1, \ \dot{q}_2)| \tag{1.112}$$

where F is shown in Figure 2 and is specified by the constraints

$$| \dot{q}_1 | \leq \dot{q}_{1o}, \tag{1.113}$$

$$| \dot{q}_2 | \leq \dot{q}_{2o}. \tag{1.114}$$

The maximum of σ_1, required in equation (1.112), will occur at a point $\dot{\mathbf{q}} \in F$ which is either inside F or on the boundaries of F where one or both constraints might be active. Furthermore, because of property 3 of the quadratic mapping, we only need to look at the half-set $K_2J_1K_1$ and its boundaries J_1K_1 and J_1K_2. Therefore, to obtain the maximum of (1.110) under the constraints (1.113) and (1.114), we should consider the following possibilities.

1. Neither of the constraints is active, i.e., $\max \sigma_1(\dot{q}_1, \dot{q}_2)$ occurs at a point $\dot{\mathbf{q}}$ inside F.

2. Constraint (1.113) is active, i.e., $\max \sigma_1(\dot{q}_1, \dot{q}_2)$ occurs at a point $\dot{\mathbf{q}}$ lying on the boundary J_1K_1 of F.

3. Constraint (1.114) is active, i.e., $\max \sigma_1(\dot{q}_1, \dot{q}_2)$ occurs at a point $\dot{\mathbf{q}}$ lying on the boundary J_1K_2 of F.

4. Both constraints are active, i.e., $\max \sigma_1(\dot{q}_1, \dot{q}_2)$ occurs at either (a) point J_1 or (b) point K_1 or (c) point K_2. Since K_1 and K_2 have the same image we only need to consider K_1.

To obtain the conditions for each one of the above cases to yield a maximum of σ_1, we first differentiate $\sigma_1(\dot{q}_1, \dot{q}_2)$ (equation (1.87) with respect to $\dot{q}_1$ and $\dot{q}_2$ to obtain

$$\frac{\partial \sigma_1}{\partial \dot{q}_1} = 2(a_{22}b_{11} - a_{12}b_{21})\dot{q}_1 + 2(a_{22}b_{12} - a_{12}b_{22})\dot{q}_1, \tag{1.115}$$

$$\frac{\partial \sigma_1}{\partial \dot{q}_2} = 2(a_{22}b_{12} - a_{12}b_{22})(\dot{q}_1 + \dot{q}_2). \tag{1.116}$$

Now we consider each case.

Case 1

To obtain the required $\dot{\mathbf{q}}$, we set

$$\frac{\partial \sigma_1}{\partial \dot{q}_1} = 0 \text{ and } \frac{\partial \sigma_1}{\partial \dot{q}_2} = 0. \tag{1.117}$$

We therefore set each of the right-hand sides of (1.115) and (1.116) to zero to obtain

$$\dot{q}_1 = \dot{q}_2 = 0 \tag{1.118}$$

which actually corresponds to the minimum value of $\sigma_1(\dot{q}_1,\ \dot{q}_2)$, *viz*, 0. Therefore, $\max(\sigma_1(\dot{q}_1,\ \dot{q}_2))$ does not occur at a point $\dot{\mathbf{q}}$ inside F.

Case 2

Since constraint (1.113) is active on the boundary J_1K_1 of F, we have

$$\dot{q}_1 = \dot{q}_{1o} \text{ (constant).} \tag{1.119}$$

To obtain the maximum, we set $\partial\sigma_1/\partial\dot{q}_2 = 0$. We therefore set the right-hand side of (1.116) to zero to obtain

$$\dot{q}_1 + \dot{q}_2 = 0. \tag{1.120}$$

Combining (1.119) and (1.120), we obtain

$$\dot{q}_2 = -\dot{q}_{1o} \tag{1.121}$$

and

$$\max[\sigma_1(\dot{q}_1,\ \dot{q}_2)] = \sigma_1(\dot{q}_{1o},\ -\dot{q}_{1o}). \tag{1.122}$$

Case 3

Since constraint (1.114) is active on the boundary J_1K_2 of F, we have

$$\dot{q}_2 = \dot{q}_{2o} \text{ (constant)}. \tag{1.123}$$

To obtain the maximum, we set $\partial\sigma_1/\partial\dot{q}_1 = 0$. We therefore set the right-hand side of (1.115) to zero and set $\dot{q}_2 = \dot{q}_{2o}$ to obtain the linear equation

$$(a_{22}b_{11} - a_{12}b_{21})\dot{q}_1 + (a_{22}b_{12} - a_{12}b_{22})\dot{q}_2 = 0. \tag{1.124}$$

Combining (1.123) and (1.124), we obtain the solution of the equation (1.124)

$$\dot{q}_1 = \dot{q}'', \tag{1.125}$$

where

$$\dot{q}'' = \frac{a_{22}b_{12} - a_{12}b_{22}}{a_{22}b_{11} - a_{12}b_{21}}\,\dot{q}_{2o}. \tag{1.126}$$

Therefore,

$$\max[\sigma_1(\dot{q}_1,\ \dot{q}_2)] = \sigma_1(\dot{q}'',\ \dot{q}_{2o}). \tag{1.127}$$

Case 4-a

When both constraints (1.113) and (1.114) are active, and $\max\sigma_1(\dot{q}_1,\ \dot{q}_2)$ occurs at $J_1(\dot{q}_{1o},\ \dot{q}_{2o})$, then

$$\max[\sigma_1(\dot{q}_1,\ \dot{q}_2)] = \sigma_1(\dot{q}_{1o},\ \dot{q}_{2o}). \tag{1.128}$$

Case 4-b

When both constraints (1.113) and (1.114) are active, and $\max\sigma_1(\dot{q}_1,\ \dot{q}_2)$ occurs at $K_1(\dot{q}_{1o},\ -\dot{q}_{2o})$, then

$$\max[\sigma_1(\dot{q}_1,\ \dot{q}_2)] = \sigma_1(\dot{q}_{1o},\ -\dot{q}_{2o}). \tag{1.129}$$

Therefore, $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1)$ is obtained as the maximum of four quantities defined by equations (1.122), (1.127), (1.128), and (1.129). In exactly analogous fashion, $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_2)$ is obtained as in (1.92).

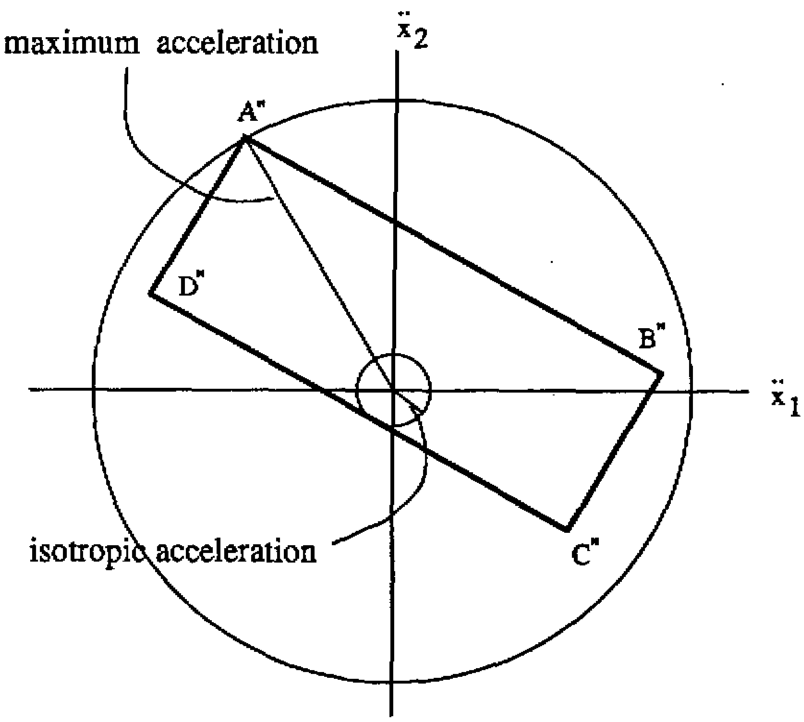

Figure 15: Characterization of the state acceleration set of a two degree-of-freedom manipulator

5.3 Properties of the state acceleration set S_u

Definition:

K : centroid of the acceleration set in the $\ddot{x}$ - plane with coordinates k_1, k_2 given by (1.38).

$\rho(K,\ l_1)$: distance from point K to the reference line l_1.

$\rho(K,\ l_2)$: distance from point K to the reference line l_2.

$\rho(A'B'),\ \rho(A''B''), \ldots$: distance from the origin to $A'B', A''B'', \ldots$ (see Figure 16)

Result 1: *The maximum acceleration corresponding to any dynamic state* u *of the manipulator is denoted* by $a_{max}(S_u)$ and is given by

$$a_{max}(S_u) = \max[d(OA''),\ d(OB''),\ d(OC''),\ d(OD'')] \tag{1.130}$$

where

$$
\begin{aligned}
d(OA'') &= \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} + k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} + k_1)^2} \\
d(OB'') &= \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} + k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} + k_1)^2}, \\
d(OC'') &= \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} - k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} - k_1)^2}, \\
d(OD'') &= \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} - k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} - k_1)^2}.
\end{aligned}
$$

Result 2: The necessary and sufficient conditions for the existence of the isotropic acceleration are the following:

$$|\det \mathbf{A}|\tau_{1o} - |k_1 a_{22} - k_2 a_{12}| > 0, \tag{1.131}$$

$$|\det \mathbf{A}|\tau_{2o} - |k_1 a_{21} - k_2 a_{11}| > 0. \tag{1.132}$$

Result 3: The isotropic acceleration corresponding to any dynamic state $\mathbf{u}$ of the manipulator is denoted by $a_{iso}(S_{\mathbf{u}})$ and, if conditions (1.131) and (1.132) are satisfied, is given by

$$a_{iso}(S_{\mathbf{u}}) = \min\left[\frac{|\det(\mathbf{A})|\tau_{1o} - |a_{22}k_1 - a_{12}k_2|}{\sqrt{a_{12}^2 + a_{22}^2}},\ \frac{|\det(\mathbf{A})|\tau_{2o} - |a_{21}k_1 - a_{11}k_2|}{\sqrt{a_{11}^2 + a_{21}^2}}\right]. \tag{1.133}$$

Proof of Result 1:

Let $d(OA'')$ through $d(OD'')$ denote, respectively, the distances of vertices A'' through D'' from the origin O in the $\ddot{\mathbf{x}}$ - plane. Then $a_{max}(S_{\mathbf{u}})$ is the distances of the furthest vertex of the set $S_{\mathbf{u}}$ which is the parallelogram $A''B''C''D''$. Therefore, $a_{max}(S_{\mathbf{u}})$ is given by

$$a_{max}(S_{\mathbf{u}}) = \max[d(OA''),\ d(OB''),\ d(OC''),\ d(OD'')]. \tag{1.134}$$

Using (1.40), the coordinates $\ddot{x}_1(A'')$ and $\ddot{x}_2(A'')$ of vertex A'' in the $\ddot{\mathbf{x}}$ - plane are given by

$$\ddot{x}_1(A'') = \ddot{x}_1(A') + k_1 = a_{11}\tau_{1o} + a_{12}\tau_{2o} + k_1, \tag{1.135}$$

$$\ddot{x}_2(A'') = \ddot{x}_2(A') + k_2 = a_{21}\tau_{1o} + a_{22}\tau_{2o} + k_2. \tag{1.136}$$

The distance $d(OA'')$ from the origin O to the point A'' is given by

$$
\begin{aligned}
d(OA'') &= \sqrt{\ddot{x}_1^2(A'') + \ddot{x}_2^2(A'')} \\
&= \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} + k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} + k_1)^2}.
\end{aligned} \tag{1.137}
$$

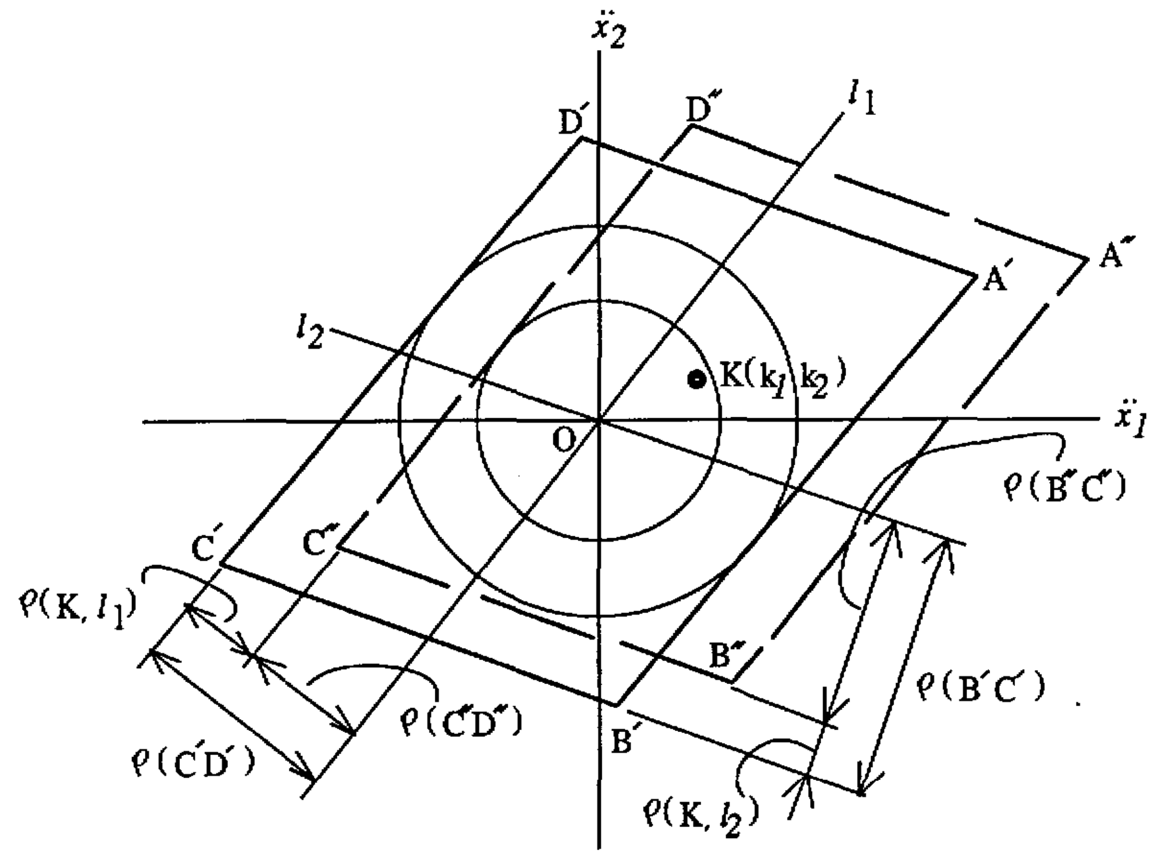

Figure 16: Isotropic acceleration of the state acceleration set of a two degree-of-freedom manipulator

In exactly analogous fashion, we obtain

$$d(OB'') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} + k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} + k_1)^2}, \tag{1.138}$$

$$d(OC'') = \sqrt{(a_{11}\tau_{1o} + a_{12}\tau_{2o} - k_1)^2 + (a_{21}\tau_{1o} + a_{22}\tau_{2o} - k_1)^2}, \tag{1.139}$$

$$d(OD'') = \sqrt{(a_{11}\tau_{1o} - a_{12}\tau_{2o} - k_1)^2 + (a_{21}\tau_{1o} - a_{22}\tau_{2o} - k_1)^2}. \tag{1.140}$$

Equations (1.134) and (1.137) through (1.140) comprise result 1.

Proof of Result 2 and 3:

In Figure 16, we have shown two sets, S_τ and $S_{\mathbf{u}}$ which is obtained from S_τ by a translation $k = (k_1,\ k_2)^T$. The centroids of S_τ and $S_{\mathbf{u}}$ are, respectively, by O and K.

Using equations (1.79), (1.64), (1.45), and (1.46), the distance from K to the reference lines l_1 and l_2 are given by

$$\rho(K,\ l_1) = \frac{|a_{22}k_1 - a_{12}k_2|}{\sqrt{a_{22}^2 + a_{12}^2}}. \tag{1.141}$$

$$\rho(K,\ l_2) = \frac{|a_{21}k_1 - a_{11}k_2|}{\sqrt{a_{21}^2 + a_{11}^2}}, \tag{1.142}$$

$\rho(K,\ l_1)$ represents the perpendicular distance between the lines $A'B'$ and $A''B''$ and also between the lines $C'D'$ and $C''D''$ (see Figure 16). Similarly, $\rho(K,\ l_2)$ is equal to the perpendicular distance between the lines $B'C'$ and $B''C''$ and also between the lines $D'A'$ and $D''A''$ (see Figure 16).

The state isotropic acceleration $a_{iso}(S_u)$ is the maximum acceleration which is available in all directions. It is therefore equal to the minimum of the distances from the origin O (of the acceleration plane) to the four sides of $A''B''C''D''$ (the set S_u).

Referring to Figure 16, we can write the following expression for $a_{iso}(S_u)$:

$$a_{iso}(S_u) = \min[\rho(A''B''),\ \rho(B''C''),\ \rho(C''D''),\ \rho(D''A'')] \tag{1.143}$$

where $\rho(A''B'')$ is the (perpendicular) distance from O to $A''B''$ and similarly for $\rho(B''C'')$, $\rho(C''D'')$, $\rho(D''A'')$, all assumed positive by definition. from the geometry of Figure 16, we can write,

$$\rho(A''B''), \rho(C''D'') = \rho(A'B') \pm \rho(K,\ l_1). \tag{1.144}$$

(Comment: In Figure 16, for example, $\rho(A''B'') = \rho(A'B') + \rho(K, l_1)$ and $\rho(C''D'') = \rho(C'D') - \rho(K, l_1)$; the correct choice of signs will depend on the direction of the translation but as will be shown below we do not have to worry about the correct choice of signs.)

Similarly,

$$\rho(B''C''), \rho(D''A'') = \rho(B'C') \pm \rho(K, l_2). \tag{1.145}$$

(The above comment holds for (1.145), too.)

Combining equations (1.143), (1.144), and (1.145), we obtain

$$a_{iso}(S_u) = \min[\rho(A'B') \pm \rho(K,\ l_1),\ \rho(B'C') \pm \rho(K,\ l_2)]. \tag{1.146}$$

Since all distances $\rho()$ in the above equation are positive by definition, we can rewrite the above equation as

$$a_{iso}(S_{\mathbf{u}}) = \min[\rho(A'B') - \rho(K,\ l_1),\ \rho(B'C') - \rho(K,\ l_2)]. \tag{1.147}$$

Substituting equations (1.80), (1.81), (1.141) and (1.142) into (1.147), we obtain the required result (1.133).

Equation (1.147) clearly demonstrates that the isotropic acceleration $a_{iso}(S_{\mathbf{u}})$ for any state $\mathbf{u} \neq 0$ is less than $a_{iso}(S_{\tau}) = \min[\rho(A'B'), \rho(B'C')]$. In fact, if $\rho(K,\ l_1)$ and $\rho(K,\ l_2)$ are sufficiently large (equivalently, the "nonlinearities" k_1 and k_2 are sufficiently "large"), we may not have any isotropic acceleration. The necessary and sufficient conditions for the existence of the isotropic acceleration can be obtained either from (1.147) or (1.133). From (1.133), we obtain the following two necessary and sufficient conditions for the existence of the isotropic acceleration:

$$|\det(A)|\tau_{1o} - |k_1 a_{22} - k_2 a_{12}| > 0, \tag{1.148}$$

$$|\det(A)|\tau_{2o} - |k_1 a_{21} - k_2 a_{11}| > 0. \tag{1.149}$$

These are exactly the necessary and sufficient conditions expressed in (1.131) and (1.132) of result 2.

6 Local acceleration properties

At any given (local) configuration $\mathbf{q}$ in the workspace, the following questions are of theoretical and practical importance.

- What is the magnitude of the maximum acceleration at any configuration $\mathbf{q}$ in the workspace?
- What is the magnitude of the isotropic acceleration at any configuration $\mathbf{q}$ in the workspace?

To answer both these questions, we need to use the properties of the sets S_τ, $S_{\dot{\mathbf{q}}}$ $S_{\mathbf{u}}$ developed in the preceding subsection.

Result 1: The local maximum acceleration $a_{\max,\text{local}}$ at a given configuration $\mathbf{q}$ is specified by

$$(a_{\max,\text{local}})_{\text{lb}} \leq a_{\max,\text{local}} \leq (a_{\max,\text{local}})_{\text{ub}} \tag{1.150}$$

where $(a_{\max,\text{local}})_{\text{lb}}$ is given by (1.130) with $k_1(\mathbf{q},\dot{\mathbf{q}})$ and $k_2(\mathbf{q},\dot{\mathbf{q}})$ evaluated at that joint variable vector $\dot{\mathbf{q}}$ which maximizes $l(\dot{q}_1,\ \dot{q}_2)$ in equation (1.89).

$$(a_{\max,\text{local}})_{\text{ub}} = a_{\max}(S_{\dot{\mathbf{q}}}) + a_{\max}(S_\tau) \tag{1.151}$$

where $a_{\max}(S_{\dot{\mathbf{q}}})$ is given by (1.89) and $a_{\max}(S_\tau)$ is given by (1.69).

Result 2: The local isotropic acceleration $a_{\text{iso,local}}$ at a given configuration $\mathbf{q}$ is specified by

$$\begin{aligned} & a_{\text{iso,local}} \\ = \;& \min[\rho(A'B') - \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1), \rho(B'C') - \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_2)] \end{aligned} \tag{1.152}$$

where $\rho(A'B')$ and $\rho(B'C')$ are given, respectively, by equations (1.80) and (1.81), and where $\max(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1)$ and $\max(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_2)$ are given by equation (1.92).

Proof of Result 1:

The local maximum acceleration $a_{\max}$ is the maximum acceleration over all possible state acceleration sets $S_{\mathbf{u}}$ at a given position $\mathbf{q}$ in the workspace. Therefore, $a_{\max}$ can be written as

$$a_{\max,\text{local}} = \max(\cup_{\dot{\mathbf{q}}\in F} S_{\mathbf{u}}). \tag{1.153}$$

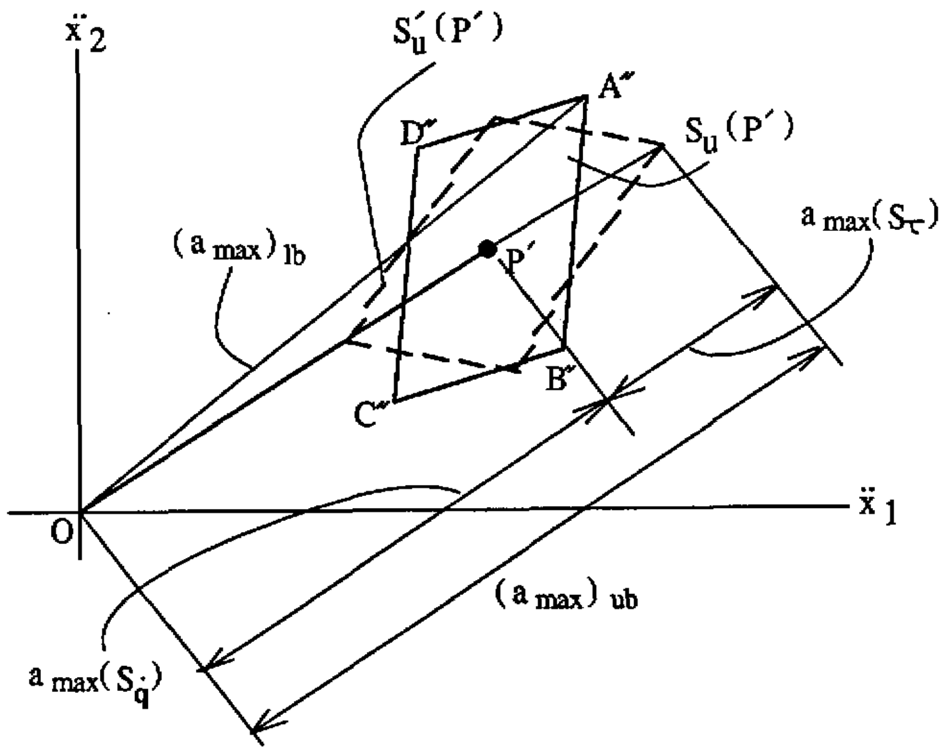

Figure 17: Maximum local acceleration of a two degree-of-freedom manipulator

It is not possible to find an exact analytical expression for $a_{\max,\text{local}}$. However, we can find an upper bound and lower bound which are very good approximations to $a_{\max,\text{local}}$.

Corresponding to every point P of the set $S_{\dot{q}}$, we have a state acceleration set $S_u(P)$. Let P' be the furthest point (from the origin) of $S_{\dot{q}}$, and let $S_u(P')$ be the corresponding state acceleration set, as shown in Figure 17. Also shown in Figure 17 is the set $S_u'(P')$ obtained by rotating the set $S_u(P')$ about P' till the longest diagonal ($A''C''$ in this case) of S_u is collinear with the line OP' joining the origin to the furthest point P' of $S_{\dot{q}}$. It is easily seen from Figure 17, that if vertex A' is the furthest vertex of $S_u(P')$ from O, then a lower bound is given by

$$(a_{\max,\text{local}})_{\text{lb}} = d(OA''), \tag{1.154}$$

and an upper bound for $a_{\max,\text{local}}$ is given by

$$(a_{\max,\text{local}})_{\text{ub}} = d(OP') + d(A''P'), \tag{1.155}$$

$$(a_{\max,\text{local}})_{\text{ub}} = a_{\max}(S_{\dot{\mathbf{q}}}) + a_{\max}(S_\tau). \tag{1.156}$$

In general, one of the four vertices A'', B'', C'', or D'' would be the furthest vertex of $S_{\mathbf{u}}$ and therefore we should write (1.154) as

$$(a_{\max,\text{local}})_{\text{lb}} = \max[d(OA''), d(OB''), d(OC''), d(OD'')]. \tag{1.157}$$

Combining (1.157) with equation (1.137) through (1.140), we obtain equation (1.130). The values of k_1 and k_2 in (1.130) correspond to the furthest vertex p' of $S_{\dot{\mathbf{q}}}$ from the origin, i.e., to that joint variable vector $\dot{\mathbf{q}}$ which maximizes $l(\dot{q}_1,\ \dot{q}_2)$ in equation (1.89). This is simply a matter of computing $l(\dot{q}_1,\ \dot{q}_2)$ at the four vectors $(\dot{q}_{1o},\ -\dot{q}_{1o})^T$, $(\dot{q}_1',\ \dot{q}_{2o})^T$, $(\dot{q}_{1o},\ \dot{q}_{2o})^T$ and $(\dot{q}_{1o},\ -\dot{q}_{2o})^T$ defined in subsection 3.2.2 and determining which of these four vectors maximizes $l(\dot{q}_1,\ \dot{q}_2)$. This completes the determination of the lower bound $(a_{\max,\text{local}})_{lb}$.

Substituting for $a_{\max}(S_{\dot{\mathbf{q}}})$ and $a_{\max}(S_\tau)$ from equations (1.89) and (1.69), respectively, we obtain equation (1.151). Thus, Result 1 is proved.

Proof of result 2:

The local isotropic acceleration is obtained in the following steps.

1. The maximum possible isotropic acceleration is obtained when $\dot{\mathbf{q}} = 0$ and is equal to $a_{\text{iso}}(S_\tau)$ as given by equation (1.70).

2. Every state acceleration set will have an isotropic acceleration which is less than that given by (1.70) because the "nonlinearities" effectively reduce the isotropic acceleration. The resulting state isotropic acceleration is $a_{\text{iso}}(S_{\mathbf{u}})$ which is given by equation (1.147).

3. The local isotropic acceleration $a_{\text{iso,local}}$ is the magnitude of the smallest state isotropic acceleration at a given local configuration $\mathbf{q}$, i.e.

$$a_{\text{iso,local}} = \min_{\dot{\mathbf{q}} \in F} a_{\text{iso}}(S_{\mathbf{u}}). \tag{1.158}$$

4. Using equation (1.147) and (1.158), we can express the local isotropic acceleration $a_{\text{iso,local}}$ as

$$\begin{aligned} a_{\text{iso,local}} &= \min_{\dot{\mathbf{q}} \in F} \min[\rho(A'B') - \rho(K, l_1),\ \rho(B'C') - \rho(K, l_2)] \\ &= \min[\min_{\dot{\mathbf{q}} \in F}\{\rho(A'B') - \rho(K, l_1)\},\ \min_{\dot{\mathbf{q}} \in F}\{\rho(B'C') - \rho(K, l_2)\}]. \end{aligned} \tag{1.159}$$

5. Since $\rho(A'B')$ and $\rho(B'C')$ are constants for a given manipulator and given actuator constraints, (1.159) can be written as

$$a_{\text{iso,local}} = \min[\rho(A'B') - \max\rho(K, l_1),\ \rho(B'C') - \max\rho(K, l_2)]. \tag{1.160}$$

where $\max(\rho(K, l_1))$ is the distance from the line l_1 to the element of $S_{\dot{\mathbf{q}}}$ furthest away from l_1 which we denoted in subsection 3.2.2 by $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1)$, and $\max(\rho(K, l_2))$ is the distance from the line l_2 to the element of $S_{\dot{\mathbf{q}}}$ furthest away from l_2 which we denoted by $\rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_2)$. We can therefore write

$$\max\rho(K, l_1) = \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_1) \tag{1.161}$$

$$\max\rho(K, l_2) = \rho_{\max}(\ddot{\mathbf{x}}(S_{\dot{\mathbf{q}}}), l_2) \tag{1.162}$$

Combining (1.160), (1.161) and (1.162), we obtain the required result (1.152). (Note that all quantities in (1.152) have been analytically determined earlier.)

7 Summary and conclusions

In this paper, we have developed a theory for the acceleration sets of planar manipulators. In particular, we have accomplished the following:

- Given the kinematical and dynamical equations of a manipulator, we have defined the image set S_τ corresponding to the set T of actuator torques, and the image set $S_{\dot{q}}$ corresponding to the set F of the joint variable rates. We have also defined the state acceleration set $S_{\mathbf{u}}$ at a specified point $\mathbf{u}$ in the state space.
- We have determined the image sets, S_τ and $S_{\dot{q}}$, and the state acceleration set $S_{\mathbf{u}}$.
- We have characterized the image sets S_τ and the state acceleration set $S_{\mathbf{u}}$ by their maximum and isotropic acceleration. The image set $S_{\dot{q}}$ has been also characterized by the maximum acceleration.
- At a configuration or position, $\mathbf{q}$, in the workspace, we have established two local acceleration properties: the local maximum acceleration and the local isotropic acceleration. The local maximum acceleration specifies the magnitude of the maximum acceleration of (a reference point on) the end-effector. The local isotropic acceleration specifies the magnitude of the maximum available acceleration of the end-effector in all directions.

We have, therefore, demonstrated the hypothesis which we stated in the introduction, i.e., that the analytical properties of acceleration sets can be determined from the properties of the linear and quadratic mappings which define them (the acceleration sets). Furthermore, the acceleration properties of interest - especially the isotropic acceleration - have been determined in terms of the manipulator parameters and the torque limits and joint variable rate ("joint velocity") limits. The stage has now been set for the application of the theory developed in this paper to problems in the design of manipulators in the companion paper (Desa and Kim, 1989).

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Appendix. Equations of motion for the two degree-of-freedom planar manipulators

1. Jacobian matrix

The joint variable rate is related to the velocity in Cartesian space by the Jacobian matrix,

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}.$$

The Jacobian matrix $\mathbf{J}$ of the two degree-of-freedom manipulator shown in Figure 1 is the following:

$$\mathbf{J} = \begin{bmatrix} -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix}$$

When this relationship is differentiated with respect to the time, we obtain the following equation,

$$\ddot{\mathbf{x}} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = \mathbf{J}\ddot{\mathbf{q}} - \mathbf{E}\{\dot{\mathbf{q}}\}^2 \qquad (1.163)$$

where $\mathbf{E}$ is the matrix which has the following elements:

$$\mathbf{E} = \begin{bmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) & l_2 \sin(q_1 + q_2) \end{bmatrix}$$

2. Dynamic equation

The dynamics of the two-degree-of-freedom planar manipulator shown in Figure 1 is described by the following equation:

$$\mathbf{D}\ddot{\mathbf{q}} + \mathbf{V}\{\dot{\mathbf{q}}\}^2 = \tau \ , \qquad (1.164)$$

where the components of matrices $\mathbf{D}$ and $\mathbf{V}$ are as follows:

$$\mathbf{D} = \begin{bmatrix} I_1 + m_1 a_1^2 + I_2 + m_2(a_2^2 + 2a_2 l_1 \cos q_2 + l_1^2) & I_2 + m_2(a_2^2 + a_2 l_1 \cos q_2) \\ I_2 + m_2(a_2^2 + a_2 l_1 \cos q_2) & I_2 + m_2 a_2^2 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 0 & -m_2 a_2 l_1 \sin q_2 \\ m_2 a_2 l_1 \sin q_2 & 0 \end{bmatrix} ,$$

and the nonlinear vector $\{\dot{\mathbf{q}}\}^2$ is as follows:

$$\{\dot{q}\}^2 = \begin{bmatrix} \dot{q}_1^2 \\ (\dot{q}_1 + \dot{q}_2)^2 - \dot{q}_1^2 \end{bmatrix}$$

3. Acceleration equation

The expression of the acceleration of the end-effector is as follows:

$$\ddot{\mathbf{x}} = \mathbf{A}\tau + \mathbf{B}\{\dot{\mathbf{q}}\}^2 \tag{1.165}$$

where

$$\mathbf{A} = \mathbf{J}\mathbf{D}^{-1} \tag{1.166}$$

$$\mathbf{B} = -\mathbf{A}\mathbf{V} - \mathbf{E} \tag{1.167}$$

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