# A Composite Discrete/Continuous Control of Robot Manipulators

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## ABSTRACT

In this report, a composite control scheme for the control of robot manipulators is proposed.

Due to the modeling error or environmental uncertainties, robot motion may present a significant positioning error by using a conventional Computer-Torque Method. To improve tracking capability of robot manipulators, sliding mode control and nonlinear control algorithms have been introduced, but computation is costly, and thus a fast motion execution using simple computer sources is impossible.

To solve this problem, we present a composite control algorithm to control robot motion combining a discrete feedforward component and a continuous feedback component. The discrete feedforward component provides a nominal torque computed using the robot dynamics and compensates for dynamic coupling between the links. This part can be updated in a large sampling time, and can be computed off-line generally, thus real time computation is decreased. The continuous feedback control component uses a structure of Variable Structure System and provides a robust control to disturbances during the sliding mode. This part can be digitally implemented using a short sampling time, and thus a fast motion of a multi-degree freedom robot manipulator can be executed by using a simple computer, or even a single board computer with an 8-bit CPU.

The stability of the proposed multiple-rate control scheme is proven in the paper and efficiency of the control scheme has been demonstrated by simulations of a three-link robot subject to parameter and payload uncertainties.

# 1 Introduction

The lack of efficient and robust real-time control algorithms for high speed motions is one of the important reasons why the applications of the present robotic manipulators are limited. The dynamic equation of robotic manipulator is highly nonlinear due to the inertia and strong coupling terms among the joints, such as centrifugal, Coriolis and gravitational forces [11, 12]. It is difficult to guarantee the tracking error bound in high speed motion, by neglecting the nonlinear dynamic terms which may act as a large disturbance to the controller. In order to improve the trajectory tracking accuracy, it is necessary to take the robot manipulator dynamics into consideration [12].

The well-known Computed Torque Method (CTM) normally provides a feasible controller if the exact knowledge of the manipulator dynamics is available. However, for a large amount of applications, it is impossible to obtain the complete dynamic model of robots, due to modeling uncertainties, parameter variation and unknown payloads. These uncertainties, especially the error of inertia matrix, may result in the instability of robot systems [17, 13]. On the other hand, the computation time of such a complex dynamics also makes its implementation impractical in some cases.

The sliding mode controller based on the Variable Structure System (VSS) method has the properties of rejection to disturbance and insensitivity to parameter variations. The method does not need to have a complete knowledge of the accurate model, and only knowledge required is the bounds of uncertain parameters of the system for the design of the controller [18]. These two features are exactly the merits for the control of a manipulator which is subjected to the modeling uncertainties and large disturbances. Therefore, the sliding mode control [10, 6, 4, 5] has been proposed in many robot control algorithms.

Depending on the side of the hyperplane (i.e., sliding surface) that the system belongs to, the VSS is of two structures. If the control structure can be switched with an ideally infinite frequency, the motion of the controlled system remains on the sliding surface. Then, the system is governed by dynamics of the sliding surface only, and the system is insensitive to parameter variations and disturbances. However, an ideal switching of the input with an infinite frequency is practically impossible due to the switching delays and neglecting time constants. Instead, the control input switches with a finite high frequency and the motion of the system is within some neighborhood of the sliding surface with chattering. This chattering is generally undesirable in practice, since it involves extremely high control activity and thereby excites the high frequency dynamics that is neglected in the model.

To solve this problem, various algorithms have been proposed to replace the discontinuous control in neighborhood of the sliding surface by the continuous control [6, 4, 5], such as the VSS algorithm, if the trajectory is outside the boundary of the sliding surface. If the trajectory is within the boundary of the sliding surface, however, a lot of people suggested to interpolate the control by proper continuous function to minimize the chattering caused by a switching input. In the implementation of these algorithms digitally, we need to compute robot model and feedback of position and velocity at every sampling time. In spite of the efficient recursive dynamic algorithms [11, 12, 8] and computing architectures [15], the computation of the model is relatively more costly. The time delay of control input, due to the computation time, deteriorates the performance in real-time control systems [9].

To reduce the time delay of control, it is desirable that the feedback is not involved in computation of the model, and the feedforward compensation using the nominal torque is good in this sense [3, 1]. The adaptive control algorithm with feedforward compensation provides a robust method to control robot manipulators. Actually there are a lot papers about the stability of the adaptive control. The feedback component of the adaptive control needs a considerable amount of computation. We believe that the composite controller combining the discrete feedforward and continuous feedback controls provides a good trajectory tracking performance in the real-time implementation.

In this paper, a composite control algorithm is proposed and the stability of the system is proven. The proposed algorithm is comprised of discrete and continuous control loops. The discrete feedforward component provides a nominal torque computed using robot dynamics and compensates for dynamic coupling between the links. This part can be updated in a large sampling time, and can be computed off-line generally, thus a real time computation is infeasible. The continuous feedback control component uses a structure of Variable Structure System and provides a robust control to disturbances of the system during the sliding mode. This part can be digitally implemented using a short sampling time, and thus a fast motion of a multi-degree freedom robot manipulator can be executed by using a simple computer, or even a single board computer with an 8-bit CPU.

The rest of the paper is organized as follows. In Section 2, we describe preliminary Lemmas as a preparation for the main control algorithms. In Section 3, we present a new composite control algorithm with proofs. In Section 4, the efficiency of the proposed algorithm for the position control is demonstrated by the simulation of a three degrees-of-freedom manipulator. The robust property to the modeling errors, the time delay of computation, parameter uncertainties and payload variations is discussed. We conclude the paper in Section 5.

## 2 Preliminaries

The motion equations of an $n$ degree-of-freedom (d.o.f.) manipulator can be derived using the Lagrange-Euler formulation as

$$D(q(t))\ddot{q}(t) + h(q(t), \dot{q}(t)) = \tau(t) \tag{1}$$

where $\tau(t) \in R^n$ is a joint input torque vector; $q(t), \dot{q}(t), \ddot{q}(t) \in R^n$ are the generalized position, velocity and acceleration vectors of the joint angles; $D(q(t)) \in R^{n\times n}$ is a symmetric positive definite inertia matrix, and $h(q(t), \dot{q}(t)) \in R^n$ is a nonlinear coupling vector including centrifugal, Coriolis and gravitational forces [2]. In the following, we denote $D(q(t))$ by $D$ and $h(q(t), \dot{q}(t))$ by $h$ for brevity.

Let us define the state vector $x(t) \in R^{2n}$ as

$$x(t) = [q(t)^T, \dot{q}(t)^T]^T. \tag{2}$$

Then the state equation of the robot system is

$$\dot{x}(t) = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -D^{-1}h \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1} \end{bmatrix} \tau(t). \tag{3}$$

Given the desired trajectories $q_d(t), \dot{q_d}(t), \ddot{q_d}(t) \in R^n$ and initial time $t_o$, we define the sliding surface vector s(t) $\in R^n$ as

$$s(t) = \dot{e}(t) + K_v e(t) + K_p \int_{t_0}^{t} e(\tau) d\tau \tag{4}$$

where $e(t) = q(t) - q_d(t)$ is an error vector in joint space and $K_v, K_p \in R^{n\times n}$ are gain matrices. For the use of the following derivation, we introduce intermediate trajectories, $q_*(t), \dot{q}_*(t), \ddot{q}_*(t) \in R^n$, which satisfies the following equation

$$\ddot{e}(t) + K_v \dot{e}(t) + K_p e(t) = 0. \tag{5}$$

Then we can rewrite (5) as follows

$$\dot{x}_*(t) - \dot{x}_d(t) = A \cdot [x_*(t) - x_d(t)] \tag{6}$$

where $x_*(t) = [q_*(t)^T, \dot{q}_*(t)^T]^T \in R^{2n}$, $x_d(t) = [q_d(t)^T, \dot{q}_d(t)^T]^T \in R^{2n}$ and $A \in R^{2n\times 2n}$ is

$$A = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix}. \tag{7}$$

Since $det[\lambda I - A] = det[\lambda^2 I + \lambda K_v + K_p]$, we can choose $K_v$ and $K_p$ so that all the eigenvalues of the matrix A have negative real parts, which guarantees the exponential stability of the system (6), and then there exist $g > 0$ and $\kappa > 0$ such that

$$\|e^{At}\| \leq g \cdot e^{-\kappa t} \tag{8}$$

for all $t \geq 0$ [16]. Here, the Euclidean matrix norm of A is defined as

$$\|A\| = [\lambda_M(A^T A)]^{\frac{1}{2}} \tag{9}$$

where $\lambda_M(\cdot)$ denotes the maximum eigenvalue of a matrix. Now, we will state the following two Lemmas as a prerequisite to the main theorems.

**Lemma 1** : If the sliding surface defined by Equation (4) satisfies $\|s(t)\| \leq \gamma$ for any $t \geq t_o$, then

$$\|x(t) - x_*(t)\| \leq [\|x(t_o) - x_*(t_o)\| + 2\gamma] \cdot e^{\|A\|t} \tag{10}$$

is satisfied for all $t \geq t_o$.

**Proof** : Using (4) and (6), we may rewrite (3) as

$$\dot{x}(t) - \dot{x}_*(t) = A \cdot [x(t) - x_*(t)] + \begin{bmatrix} 0 \\ \dot{s}(t) \end{bmatrix}. \tag{11}$$

Integrating both sides of (11) yields

$$[x(t) - x_*(t)] = [x(t_o) - x_*(t_o)] - \int_{t_o}^{t} A \cdot [x(\eta) - x_*(\eta)] d\eta + \begin{bmatrix} 0 \\ s(t) - s(t_o) \end{bmatrix}. \tag{12}$$

Taking the norm of both sides, we get

$$\|x(t) - x_*(t)\| \leq [\|x(t_o) - x_*(t_o)\| + 2\gamma] + \int_{t_o}^{t} \|A\| \cdot \|x(\tau) - x_*(\tau)\| d\tau. \tag{13}$$

If we apply the Bellman-Gronwall Inequality [14] to (13), we obtain

$$\|x(t) - x_*(t)\| \leq [\|x(t_o) - x_*(t_o)\| + 2\gamma] \cdot e^{\|A\|t} \tag{14}$$

for all $t \geq t_o$, and thus Lemma 1 is true.

If we take $x_*(t_o) = x(t_o)$ at the initial time, Equation (10) becomes

$$\|x(t) - x_*(t)\| \leq 2\gamma \cdot e^{\|A\|t}. \tag{15}$$

The above Lemma implies that the distance from real trajectory $x(t)$ to the intermediate trajectory $x_*(t)$ is bounded for a finite time. The Lemma 2 below shows that the boundness of the tracking error holds also for an infinite time interval. Considering $\rho$ as a positive number and vector $v \in R^n$, we define the neighborhood set as follows

$$S(\rho; v) = \{w \in R^n; \|w - v\| \leq \rho\}. \tag{16}$$

**Lemma 2** : Suppose $\|s(t)\| \leq \gamma$ is satisfied for all $t \geq t_o$ for some $t_o$, and the system (6) is exponentially stable and satisfies (8). Then $x(t)$ converges exponentially into the set $S(\epsilon\gamma; x_d(t))$ with $\epsilon$ which is given by

$$\epsilon = 2 \cdot (1 + \mu) \cdot \{\frac{\mu}{g \cdot (1 + \mu)}\}^{-\mu} \tag{17}$$

where $\mu$ is defined as

$$\mu = \frac{\|A\|}{\kappa}. \tag{18}$$

**Proof** : *Since the intermediate trajectory satisfies* (8), there exists $T = T(\alpha) < \infty$

$$T = -\frac{\ln(\alpha/g)}{\kappa} \tag{19}$$

which yields

$$\|x_*(t+T) - x_d(t+T)\| \leq \alpha \cdot \|x_*(t) - x_d(t)\| \tag{20}$$

for any $\alpha \in (0,1)$ and for all t. Define $E(\alpha, \beta)$ as

$$E(\alpha, \beta) = \frac{2 \cdot e^{\|A\| \cdot T}}{\beta - \alpha} \tag{21}$$

for any $\beta \in (\alpha, 1)$. Now, if $\|x(t_o) - x_d(t_o)\| > E(\alpha, \beta) \cdot \gamma$ holds for some $t_o$, let $x_*(t_o) = x(t_o)$. Using the triangle inequality, we obtain

$$\|x(t_o+T) - x_d(t_o+T)\| \leq \|x(t_o+T) - x_*(t_o+T)\| + \|x_*(t_o+T) - x_d(t_o+T)\|. \tag{22}$$

Recalling the inequalities (15) and (20), we can rewrite (22) as follows

$$\begin{aligned} \|x(t_o+T) - x_d(t_o+T)\| &\leq 2\gamma \cdot e^{\|A\| \cdot T} + \alpha \cdot \|x_*(t_o) - x_d(t_o)\| \\ &< \{\frac{2 \cdot e^{\|A\| \cdot T}}{E(\alpha, \beta)} + \alpha\} \cdot \|x(t_o) - x_d(t_o)\| \\ &= \beta \cdot \|x(t_o) - x_d(t_o)\|. \end{aligned} \tag{23}$$

Now if $\|x(t) - x_d(t)\| > E(\alpha, \beta) \cdot \gamma$ holds for $t = t_o + T$, we repeat the previous process with the initial condition $x_*(t_o+T) = x(t_o+T)$ at time $t = t_o + T$. Then

$$\begin{aligned} \|x(t_o+2T) - x_d(t_o+2T)\| &< \beta \cdot \|x(t_o+T) - x_d(t_o+T)\| \\ &< \beta^2 \cdot \|x(t_o) - x_d(t_o)\|. \end{aligned} \tag{24}$$

If this process is repeated $n$ times, we have

$$\|x(t_o+nT) - x_d(t_o+nT)\| < \beta^n \cdot \|x(t_o) - x_d(t_o)\|. \tag{25}$$

Equation (25) implies the exponential convergence of $x(t)$ into the set $S(E(\alpha, \beta); x_d(t))$.

It remains to find out the supremum of the trajectory errors for all $\alpha$ and $\beta$ with the constraint $0 < \alpha < \beta < 1$. Clearly, the infimum of $E(\alpha, \beta)$ with respect to $\beta$ occurs as $\beta \to 1$, and this results in

$$E(\alpha, 1) = \frac{2 \cdot e^{\|A\| \cdot T}}{1 - \alpha}. \tag{26}$$

We differentiate $E(\alpha, 1)$ with respect to $\alpha$ and let the derivative be equal to 0, i.e.,

$$\frac{dE(\alpha,1)}{d\alpha} = \frac{2(1/g)(\alpha/g)^{-(\frac{\|A\|}{\kappa}+1)} \cdot \{(1+\frac{\|A\|}{\kappa})\alpha - \frac{\|A\|}{\kappa}\}}{(1-\alpha)^2} = 0. \tag{27}$$

We can obtain the minimum of $E(\alpha, 1)$ when

$$\alpha = \alpha^* \equiv \frac{\frac{\|A\|}{\kappa}}{1 + \frac{\|A\|}{\kappa}},$$

and the upper bound of the trajectory error, $\epsilon = E(\alpha^*, 1)$, is given by (17) and (18). This completes the proof of Lemma 2.

Note that the equivalent trajectory is only a virtual intermediate function between $x(t)$ and $x_d(t)$ and does not exist in a real system. If $\|x(t_o) - x_d(t_o)\| \leq \epsilon \cdot \gamma$ is satisfied at the initial time $t = t_o$, then the system trajectory satisfies $\|x(t) - x_d(t)\| \leq \epsilon \cdot \gamma$ for all $t > t_o$. In the next section, we will propose a controller which guarantees $\|s(t)\| \leq \gamma$, then the trajectory error is bounded in virtue of Lemma 2.

# 3 Composite Control Algorithm

To compute the input torque using the nonlinear control algorithms, we need the dynamic model of a robot system. For many cases an exact model is impossible due to the parameter uncertainties and payloads variations. Therefore, we express the model of the robot system (1) as follows

$$\hat{D}(q(t)) \cdot \ddot{q}(t) + \hat{h}(q(t), \dot{q}(t)) = \tau(t) \tag{28}$$

where $\hat{D}(q(t)) \cdot \ddot{q}(t)$ and $\hat{h}(q(t)), \dot{q}(t))$ represent the corresponding terms of the real system (1) with modeled values of the parameters.

In general, we may consider the control as two parts, a feedforward term and a feedback term, i.e.,

$$\tau(t) = \tau_{ff}(t) + \tau_c(t) \tag{29}$$

where

$$\tau_{ff}(t) = \hat{D}(\bar{q}_d(t))\bar{\ddot{q}}_d(t) + \hat{h}(\bar{q}_d(t), \bar{\dot{q}}_d(t)) \tag{30}$$

and

$$\tau_c(t) = \hat{D}(\bar{q}_d(t)) \cdot u_c(t) \tag{31}$$

$$u_c(t) = -K_v\dot{e}(t) - K_p e(t) - k_s s(t) - k_o \frac{s(t)}{\|s(t)\| + \Delta}. \tag{32}$$

The feedforward term $\tau_{ff}(t)$ and the premultiplying coefficient, $\hat{D}(\bar{q}_d(t))$ of the feedback term $\tau_c(t)$ can be computed in off-line, and thus are step functions with time interval $T_\beta$ which is supposedly greater than the necessary computation time for the model (28). In spite of the development of various algorithms and enhancements of computing architectures, computation of the model is still costly and thus the required time interval $T_\beta$ is still not small. We used the notations $\bar{q}_d(t), \bar{\dot{q}}_d(t), \bar{\ddot{q}}_d(t)$ to denote the sampled values of the corresponding desired trajectories $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ with a sampling internal $T_\beta$ (i.e., $\bar{q}_d(t) = q_d(kT_\beta)$, $\bar{\dot{q}}_d(t) = \dot{q}_d(kT_\beta)$, and $\bar{\ddot{q}}_d(t) = \ddot{q}_d(kT_\beta)$ for all $t \in [kT_\beta, (k+1)T_\beta)$). For brevity, $\hat{D}(\bar{q}_d(t))$ is denoted by $\hat{D}_d$ and $\hat{h}(\bar{q}_d(t), \bar{\dot{q}}_d(t))$ by $\hat{h}_d$ . If the robot system (1) is controlled by the input torque computed by (29)-(32), we obtain

$$\dot{s}(t) = -k_s s(t) - k_o D^{-1}\hat{D}_d \frac{s(t)}{\|s(t)\| + \Delta} + n(t) \tag{33}$$

where the disturbance vector, $n(t) \in R^n$, is given by

$$\begin{aligned} n(t) &= n(q_d, \dot{q}_d, \ddot{q}_d, q, \dot{q}) \\ &= \delta D(t) \cdot [\bar{\ddot{q}}_d(t) - K_v\dot{e}(t) - K_p e(t) - k_s s(t)] + [\bar{\ddot{q}}_d(t) - \ddot{q}_d(t)] + D^{-1} \cdot [\hat{h}_d - h] \end{aligned} \tag{34}$$

and $\delta D(t)$ is defined as

$$\delta D(t) = D^{-1}\hat{D}_d - I. \tag{35}$$

We can rewrite (33) as

$$\dot{s}(t) = -k_s s(t) - k_o \frac{s(t)}{\|s(t)\| + \Delta} - k_o \frac{\delta D(t) \cdot s(t)}{\|s(t)\| + \Delta} + n(t). \tag{36}$$

Define the constants $N$ and $M$ as such

$$N = \max_{t,z,w}\{\|n(q_d, \dot{q}_d, \ddot{q}_d, z, w)\|; [z(t)^T, w(t)^T]^T \in S(\epsilon\gamma; x_d(t))\} \tag{37}$$

$$M = \max_{t,z}\{\|\delta D(q_d, z)\|; z(t) \in S(\epsilon\gamma; q_d(t))\}. \tag{38}$$

In what follows, we prove the stability of the system using the controller given by (29)-(32).

**Theorem 1** : For a robotic system (1) using controller given by (29)- (32), if at the initial time $t = t_o$ for any $\gamma > 0$,

$$\|s(t_o)\| \leq \gamma \quad , \quad \|x(t_o) - x_d(t_o)\| \leq \epsilon \cdot \gamma,$$

and the gain $k_o$ is bounded by

$$N \cdot (1 + \frac{\Delta}{\gamma}) < k_o < \frac{(k_s - \rho) \cdot (\Delta + \gamma)}{M} \tag{39}$$

for the given $K_v$, $K_p$, $k_s$ and for a small positive $\rho$, then the system tracking error satisfies

$$\|s(t)\| \leq \gamma \quad , \quad x(t) \in S(\epsilon\gamma; x_d(t)) \tag{40}$$

for all $t \geq t_o$.

**Proof** : We consider $V(t) = \frac{1}{2}s(t)^T s(t)$ as a Lyapunov function and differentiate it with respect to $t$,

$$\frac{dV}{dt} = s(t)^T \dot{s}(t) = \{s(t)^T n(t) - k_o \frac{s(t)^T s(t)}{\|s(t)\| + \Delta}\}$$

$$+ \{-k_o \frac{s(t)^T \cdot \delta D(t) \cdot s(t)}{\|s(t)\| + \Delta} - k_s s(t)^T s(t)\}. \tag{41}$$

Using the matrix inequality [7],

$$x^T A y \leq \|x\| \cdot \|y\| \cdot \|A\| \tag{42}$$

we obtain

$$\frac{dV}{dt} \leq \|s(t)\|\{\|n(t)\| - k_o \frac{\|s(t)\|}{\|s(t)\| + \Delta}\} + \|s(t)\|^2\{k_o \frac{\|\delta D(t)\|}{\|s(t)\| + \Delta} - k_s\}. \tag{43}$$

If we assume that $x(t) \notin S(\epsilon\gamma; x_d(t))$ for some $t = t_2$, there exists $t_1 \in [t_o, t_2)$ such that $s(t) \in S(\gamma)$ for all $t \in [t_o, t_1)$ and $\|s(t_1)\| = \gamma$, since $\|x(t_o) - x_d(t_o)\| \leq \epsilon \cdot \gamma$ and the motion trajectory is continuous. Then $x(t_1) \in S(\epsilon\gamma; x_d(t_1))$ is satisfied (from Lemma 2), and $\|n(t_1)\| \leq N$ and $\|\delta D(t_1)\| \leq M$ are satisfied from the definitions of (37) and (38). Hence, we can rewrite (43) as

$$\frac{dV}{dt} \leq \|s(t)\|\{N - k_o \frac{\|s(t)\|}{\|s(t)\| + \Delta}\} + \|s(t)\|^2\{k_o \frac{M}{\|s(t)\| + \Delta} - k_s\} \tag{44}$$

at $t = t_1$. If the gain $k_o$ satisfies the condition (39), then

$$\frac{dV}{dt}(t) < -\rho \cdot s(t)^T s(t) \tag{45}$$

at $t = t_1$, which contradicts the assumption that $x(t) \notin S(\epsilon\gamma; x_d(t))$. This completes the proof of Theorem 1.

If we use a small sampling time $T_\beta$ and have a relatively accurate model, then the maximum values of N and M can be small. In this case, the lower bound of $k_o$ can be made to be small and the upper bound of $k_o$ can be made to be large enough to ensure the existence of the gain $k_o$. Since the role of feedforward component $\tau_{ff}(t)$ is compensation for the dynamics and nonlinear coupling torques between the joints, we may take a large sampling time $T_\beta$ for the discrete terms $\hat{D}_d$ and $\hat{h}_d$ to reduce the computation. Of course, the sampling time should not be too large. With increase of the sampling time, the magnitude of modeling errors, $\|n(t)\|$ and $\|\delta D(t)\|$ may become large, and thus the required bounds of the gain $k_o$ become severe and the trajectory error is increased. This will be discussed in detail in Section 4.

Using the smaller $\gamma$, $s(t)$ remains closer to the surface $s(t) = 0$ and the trajectory error becomes smaller. In this case, the lower bound of the gain $k_o$ is not necessarily increased since the constant $N$ of (37) is decreased. If we take the smaller $\Delta$ in the control algorithm (32), then both the lower and upper bounds of the gain $k_o$ are decreased. However, it is difficult to expect that the control changes smoothly.

In order to achieve a smooth change of control output, we may consider the interpolation of the discrete terms $\hat{D}_d$ and $\hat{h}_d$ in (30) and (31). Since these terms are functions of the desired trajectories and can be computed in off-line, the interpolation of these terms can be achieved by various simple methods.

Substituting the continuous feedback terms $q(t)$ and $\dot{q}(t)$ to the sampled nominal trajectory $\bar{q}_d(t)$ and $\dot{\bar{q}}_d(t)$, we may get the continuous control input in a combined form as follows

$$\tau(t) = \hat{D}(q(t)) \cdot u(t) + \hat{h}(q(t), \dot{q}(t)) \tag{46}$$

$$u(t) = \ddot{q}_d(t) - K_v\dot{e}(t) - K_p e(t) - k_s s(t) - k_o \frac{s(t)}{\|s(t)\| + \Delta}. \tag{47}$$

Then, we can prove the boundedness of the tracking errors of the system (1) by the controller (46) and (47) in the same manner as in Theorem 1.

If the model is relatively accurate and the sampling time of discrete components ($T_\beta$) is small, it is not so severe to assume that $M < 1$. In this case, we can obtain the following Corollary which gives another condition for $k_o$.

**Corollary 1** : Consider the robot system (1) using the controller given by (29)-(32). If $M < 1$ and at the initial time $t = t_o$, $\|s(t_o)\| \le \gamma$ and $\|x(t_o) - x_d(t_o)\| \le \epsilon \cdot \gamma$ for any $\gamma > 0$ is satisfied at the initial time $t_o$, and if the gain $k_o$ is bounded by

$$\frac{N \cdot (1 + \frac{\Delta}{\gamma})}{1 - M} < k_o \tag{48}$$

for given $K_v$, $K_p$ and positive $k_s$, then the system tracking error satisfies

$$\|s(t)\| \leq \gamma, \quad x(t) \in S(\epsilon\gamma; x_d(t)) \tag{49}$$

for all $t \geq t_o$.

**Proof** : We consider $V(t) = \frac{1}{2}s(t)^T s(t)$ as a Lyapunov function and differentiate it with respect to $t$,

$$\frac{dV}{dt} \leq \|s(t)\|\{\|n_1(t)\| - k_o \frac{\|s(t)\|}{\|s(t)\| + \Delta} \cdot [1 - \|\delta D(t)\|]\} - k_s s(t)^T s(t). \tag{50}$$

If we assume that $\|s(t)\| \leq \gamma$ for all $t_o \leq t \leq t_1$, and $\|s(t_1)\| = \gamma$ for some $t_1$, then it yields

$$\frac{dV}{dt} \leq \|s(t)\|\{N - k_o \frac{\|s(t)\|}{\|s(t)\| + \Delta} \cdot [1 - M]\} - k_s s(t)^T s(t) \tag{51}$$

at $t = t_1$. Hence, if the gain $k_o$ satisfies the condition (48),

$$\frac{dV}{dt} < -k_s s(t)^T s(t) \tag{52}$$

at $t = t_1$ which completes the proof of Corollary 1.

The Corollary 1 also gives some insight of the role of the term, $-k_s s(t)$, in the feedback component. Combining Theorem 1 and Corollary 1, the sufficient condition for the stability of the system with controller (29)-(32) is that $k_o$ satisfies the lower and upper bound conditions given by (39), or $k_o$ satisfies the lower bound condition given by (48) if $M < 1$.

If we eliminate the norm of $s(t)$ in $u_c(t)$, we can compute $u_{c_i}(t)$ independently for each joint and the more efficient computation is possible. Thus, instead of using the feedback control (32), we propose to use the feedback component as follows,

$$u_c(t) = -K_v \dot{e}(t) - K_p e(t) - k_s s(t) - k_o \sigma(t) \tag{53}$$

where

$$\sigma(t) = [\sigma_1, \cdots, \sigma_i, \cdots, \sigma_n]^T \ , \quad \sigma_i(t) = \frac{s_i(t)}{|s_i(t)| + \Delta}. \tag{54}$$

When the gain matrices $K_v$ and $K_p$ are diagonal, we can rewrite (53) as

$$u_{c_i}(t) = -k_{v_i} \dot{e}_i(t) - k_{p_i} e_i(t) - k_s s_i(t) - k_o \sigma_i(t). \tag{55}$$

Using the controller given by (29)-(31), and (53) for the robot system (1), the differential tracking error

$$\dot{s}(t) = -k_s s(t) - k_o \sigma(t) - k_o \cdot \delta D(t) \sigma(t) + n(t). \tag{56}$$

where the disturbance vector $n(t)$ is given by (34), and its $i$-$th$ element is denoted by $n_i(t)$. The $(i,j)$-$th$ element of $\delta D(t)$ is denoted by $\delta D_{ij}(t)$ and the constants $N_\infty$ and $M_\infty$ are defined as follows

$$N_\infty = \max_{t,z,w}\{\sum_i |n_i(\ddot{q}_d, \dot{q}_d, q_d, z, w)|; [z(t)^T, w(t)^T]^T \in Z(\epsilon\gamma; x_d(t))\} \tag{57}$$

$$M_\infty = \max_{t,z}\{\sum_i \sum_j |\delta D_{ij}(q_d, z)|; z(t) \in Z(\epsilon\gamma; q_d(t))\} \tag{58}$$

where the neighborhood set $Z(\rho; v)$ is defined below, for any scalar $\rho$ and vector $v \in R^n$

$$Z(\rho; v) = \{w \in R^n; \|w - v\|_\infty \leq \rho\} \tag{59}$$

where $\|\cdot\|_\infty$ is the infinity norm of vector which is the maximum of the absolute values of its elements.

Now we study stability of the system using the controller (29)-(31) and (53).

**Theorem 2** : Consider the system (1) with controller given by (29)-(31) and (53). If, for any $\gamma > 0$, $\|s(t_o)\|_\infty \leq \gamma$ and $x(t_o) \in Z(\epsilon\gamma; x_d(t))$ at the initial time $t_o$, and the gain $k_o$ satisfies

$$N_\infty \cdot (1 + \frac{\Delta}{\gamma}) < k_o < \frac{(k_s - \rho) \cdot (\Delta + \gamma)}{M_\infty} \tag{60}$$

for the given $K_v$, $K_p$, $k_s$ and for an arbitrary small positive number $\rho$, *then the system* tracking error

$$\|s(t)\| \leq \gamma, \quad x(t) \in Z(\epsilon\gamma; x_d(t)) \tag{61}$$

for all $t \geq t_o$.

**Proof** : Consider $V(t) = \frac{1}{2}s(t)^T s(t)$ as a Lyapunov function and differentiate it with respect to $t$,

$$\frac{dV}{dt}(t) = s(t)^T \dot{s}(t) = \{s(t)^T n(t) - k_o s(t)^T \sigma(t)\} + \{-k_o s(t)^T \delta D(t)\sigma(t) - k_s s(t)^T s(t)\}. \tag{62}$$

Equation (62) can be rewritten as

$$\frac{dV}{dt}(t) = \sum_i s_i(t) n_i(t) - k_o \sum_i \frac{s_i(t)^2}{|s_i(t)| + \Delta} - k_o \sum_i \sum_j \frac{s_i(t)\delta D_{ij}(t) s_j(t)}{|s_i(t)| + \Delta} - k_s \sum_i s_i(t)^2. \tag{63}$$

which is bounded by

$$\begin{aligned} \frac{dV}{dt}(t) &\leq \{(\sum_i |s_i(t)| \cdot |n_i(t)|) - k_o \sum_i \frac{|s_i(t)|^2}{|s_i(t)| + \Delta}\} \\ &+\{k_o \sum_i \sum_j \frac{|s_i(t)| \cdot |\delta D_{ij}(t)| \cdot |s_j(t)|}{|s_i(t)| + \Delta} - k_s \sum_i |s_i(t)|^2\} \\ &\leq \|s(t)\|_\infty \{(\sum_i |n_i(t)|) - k_o \frac{\|s(t)\|_\infty}{\|s(t)\|_\infty + \Delta}\} \\ &+ \|s(t)\|_\infty^2 \{k_o \frac{[\sum_i \sum_j |\delta D_{ij}(t)|]}{\|s(t)\|_\infty + \Delta} - k_s\}. \end{aligned} \tag{64}$$

If we assume that $\|s(t)\|_\infty \geq \gamma$ is true for some $t = t_2$, there exists $t_1 \in [t_o, t_2)$ such that $\|s(t)\|_\infty \leq \gamma$ for all $t \in [t_o, t_1)$ and $\|s(t_1)\|_\infty = \gamma$ and $\frac{dV}{dt}(t_1) > 0$. Lemma 2 implies that

$x(t_1) \in Z(\epsilon\gamma; x_d(t_1))$ and thus $\|n(t_1)\| \leq N_\infty$ and $\|\delta D(t_1)\| \leq M_\infty$ are satisfied from the definition of $N_\infty$ and $M_\infty$. In this case, we can rewrite (64) as

$$\frac{dV}{dt}(t) \leq \|s(t)\|_\infty\{N_\infty - k_o\frac{\|s(t)\|_\infty}{\|s(t)\|_\infty + \Delta}\} + \|s(t)\|_\infty^2\{k_o\frac{M_\infty}{\|s(t)\|_\infty + \Delta} - k_s\} \qquad (65)$$

at $t = t_1$. Hence, when the gain $k_o$ satisfies the condition(60), we have

$$\frac{dV}{dt}(t) < -\rho\|s(t)\|_\infty^2 \qquad (66)$$

at $t = t_1$, and this completes the proof of Theorem 2.

If we assume $\Delta$ in the feedback component (55) as to be zero, then it takes the form of the sliding mode controller. The role of $\Delta$ is to change the discrete function to the continuous function.

By substituting the sampled nominal trajectory $q_d(t)$ and $\dot{q}_d(t)$ by the feedback measurement $q(t)$ and $\dot{q}(t)$ respectively, we obtain the continuous control input in a combined form as follows

$$\tau(t) = \hat{D}(q(t)) \cdot u(t) + \hat{h}(q(t), \dot{q}(t)) \qquad (67)$$

$$u(t) = \ddot{q}_d(t) - K_v\dot{e}(t) - K_p e(t) - k_s s(t) - k_o\sigma(t). \qquad (68)$$

We may also easily prove the boundedness of the tracking errors of the system (1) by using the above controller in similar way to that used in Theorem 2.

Provided that $M_\infty < 1$ is satisfied, the following Corollary gives a different bound of the gain $k_o$ from the bound in Theorem 2.

**Corollary 2** : Consider the system(1) with the controller given by (29)-(31) and (53). If, for any $\gamma > 0$, $\|s(t_o)\|_\infty \leq \gamma$ and $x(t_o) \in Z(\epsilon\gamma; x_d(t_o))$ at the initial time $t_o$, and the gain $k_o$ satisfies the following condition for the given $K_v$, $K_p$, and positive $k_s$,

$$\frac{N_\infty \cdot (1 + \frac{\Delta}{\gamma})}{1 - M_\infty} < k_o. \qquad (69)$$

then the system tracking error is bounded by $\gamma$,

$$\|s(t)\|_\infty \leq \gamma\,, \quad x(t) \in Z(\epsilon\gamma; x_d(t)) \qquad (70)$$

for all $t \geq t_o$.

**Proof** : Consider $V(t) = \frac{1}{2}s(t)^T s(t)$ as a Lyapunov function and differentiate it with respect to $t$,

$$\frac{dV}{dt}(t) \leq (\sum_i | s_i(t) | \cdot | n_i(t) |) - k_o\sum_i \frac{s_i(t)^2}{| s_i(t) | + \Delta}$$

$$-\{k_o\sum_i\sum_j \frac{| s_i(t) | \cdot | \delta D_{ij}(t) | \cdot | s_j(t) |}{| s_i(t) | + \Delta} - k_s s(t)^T s(t)\}$$

$$\leq \|s(t)\|_\infty\{(\sum_i | n_i(t) |) - k_o\frac{\|s(t)\|_\infty}{\|s(t)\|_\infty + \Delta}[1 - \sum_i\sum_j | \delta D_{ij}(t) |]\} - k_s s(t)^T s(t). \quad (71)$$

Provided that $\|s(t)\|_\infty \leq \gamma$ for all $t \in [t_o, t_1)$ and $\|s(t_1)\|_\infty = \gamma$ for any time $t_1$, then,

$$\frac{dV}{dt}(t) \leq \|s(t)\|_\infty\{N_\infty - k_o\frac{\|s(t)\|_\infty}{\|s(t)\|_\infty + \Delta}[1 - M_\infty]\} - k_s s(t)^T s(t) \quad (72)$$

at $t = t_1$. Consider the condition (69),

$$\frac{dV}{dt}(t) < -k_s s(t)^T s(t) \quad (73)$$

at $t = t_1$. This completes the proof of Corollary 2.

For the digital implementation of the proposed algorithm, the controller may take the multiple-rate structure, where the sampling time $T_\alpha$ of the feedback component is much smaller than the sampling interval $T_\beta$ of the feedforward component. The feedback component $u_c(t)$ of (32) and (53) is simple in structure and less computational time is needed. If we consider the gain matrices $K_v$ and $K_p$ to be diagonal, we may compute $u_c(t)$ for each joint independently. When $\hat{D}_d$ is computed in off-line, the number of multiplication and addition required to compute $\tau_c(t)$ is shown in the Table I. The values in the parentheses correspond to the case n = 6, and the integral term in the sliding vector is computed, as an example, in the following way.

$$int(k) = int(k-1) + \tilde{K}_p e(k) \quad (74)$$

$$s(k) = \dot{e}(k) + K_v e(k) + int(k), \quad (75)$$

where $s(k)$ is the value of the sliding vector at the $k$-$th$ sampling time ($t = kT_\alpha$ ), and $int(0) = 0$ and $\tilde{K}_p = T_\alpha \cdot K_p$.

The computation of the robot model is $132n$ multiplications and $111n$ - 4 additions where $n$ is the number of the degree-of-freedom of the manipulator, when the Recursive Newton-Euler algorithm is used. In case $n = 6$, the number of required multiplications is 792 and additions is 662. Thus the feedback component is of higher frequency up to 40-times than the feedforward component.

**Table I The Computational Cost Summary of Feedback Control Component Using the Proposed Algorithms (The Numbers within Blankets are for the Case n=6)**

| Feedback Controls | The Number of Multiplication (Division) | The Number of Addition (Subtraction) | Square Root |
|---|---|---|---|
| Theorem 1 | 2n + 6 (18) | 2n + 5 (17) | 1 |
| Theorem 2 | n + 6 (12) | n + 6 (12) | 0 |

# 4 Simulation Results

The purpose of the simulation is to show the robust property of the proposed algorithms. The simulation result is compared with that using the computed torque algorithm. A three degree-of-freedom manipulator is used as a case study shown in Figure 1.

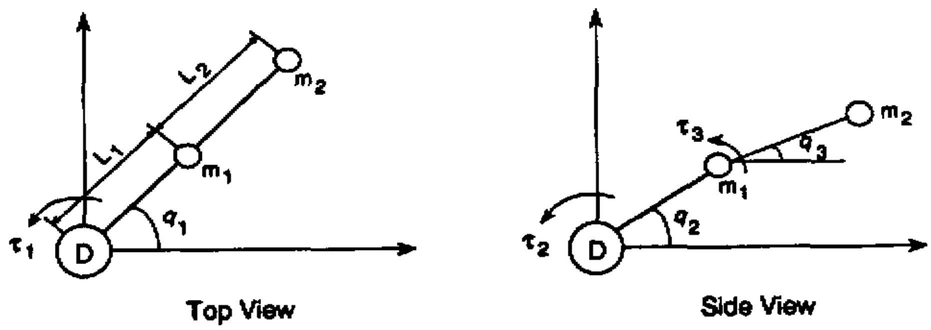



Figure 1 A Three Degree-of-Freedom Manipulator

The parameters are $D = 0.0213 kg.m$, $m_1 = m_2 = 0.782kg$, and $l_1 = l_2 = 0.23m$. To examine the robustness to modeling uncertainties, the modeling errors of each parameter, i.e., mass, length of link and moment inertia, are considered to be 1%. In simulation, the payload of 0 kg., 0.3 kg., or 0.5 kg. were carried by the manipulator. The execution time was 2 seconds and the desired trajectory was

$$q_d(t) = q_{init} + \frac{(q_{final} - q_{init}) \cdot t}{2} - \frac{(q_{final} - q_{init}) \sin(\pi t)}{2},$$

where the initial position $q_{init} = [0.4, \ -0.1, \ 0.2]^T$ (rad.), and the final position $q_{final}=$ $[-0.1, \ 0.3, \ 0.65]^T$ (rad.).

In the simulation, the sampling time $T_\alpha$ of the feedback component (32) and (53) were selected to be 1 ms. The sampling time $T_\beta$ of the feedforward component (30) can be selected to be much larger than $T_\alpha$, due to the rationale mentioned previously. Two values of $T_\beta$, 10 ms and 50 ms, is used in simulation, assuming that the time required for the computation of the model (28), $T_\gamma$, is 10 ms. The block diagram of the CTM and the proposed algorithms are shown in Figure 2 and Figure 3.

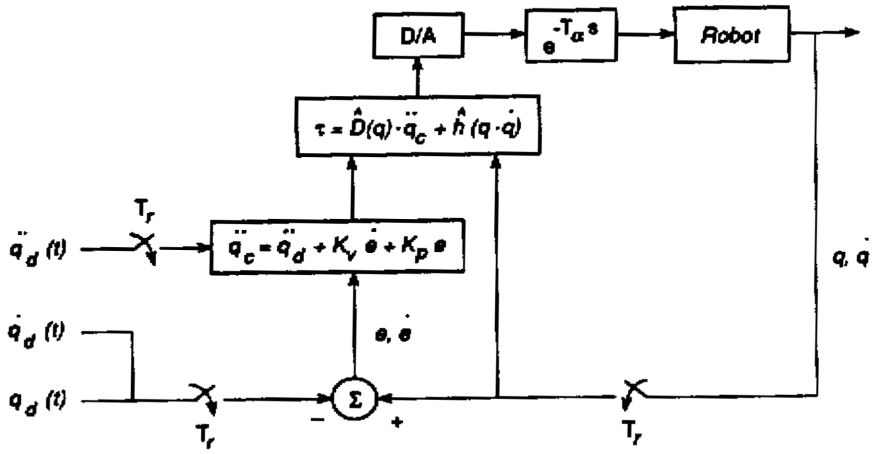



Figure 2 Block Diagram of the Computed Torque Method
($K_v$ =20 · $I$, $K_p$ =100 · $I$, $T_r$ =10 ms)

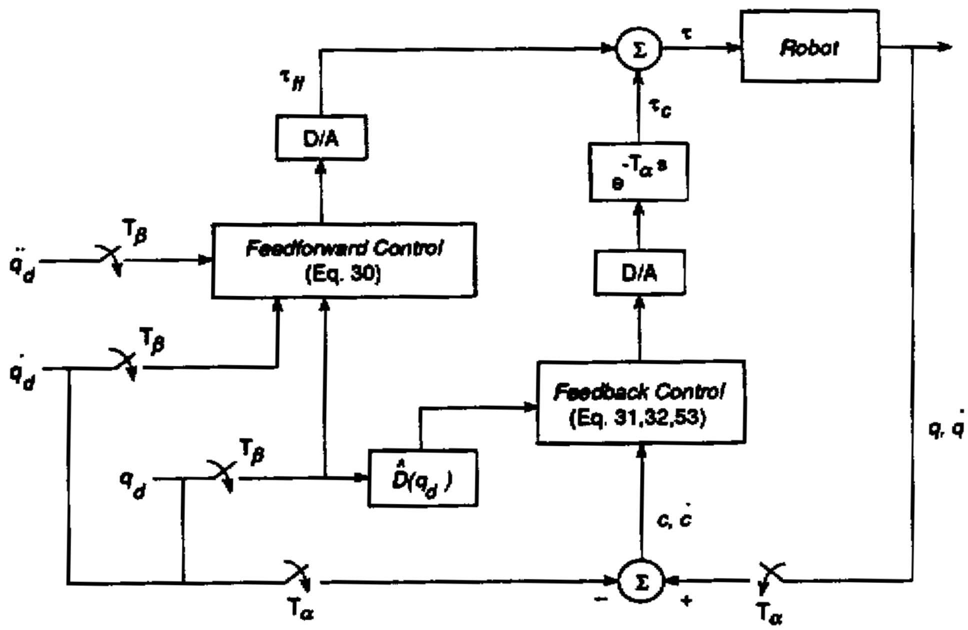
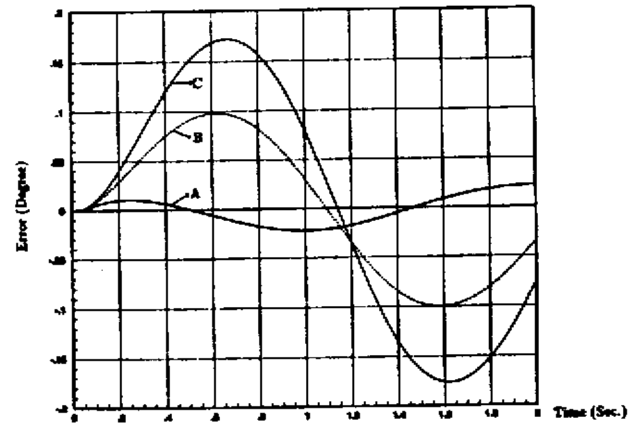
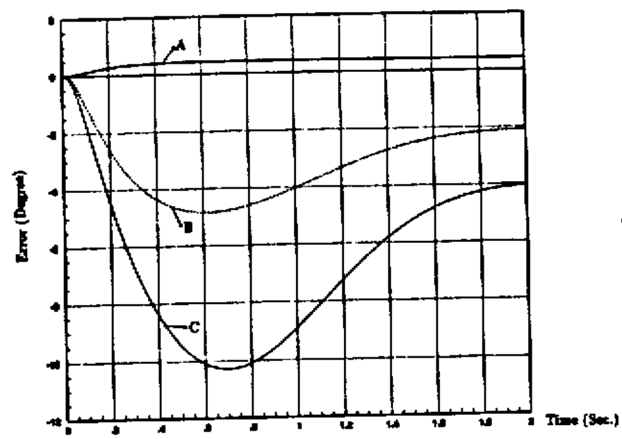



Figure 3 Block Diagram of the Proposed Control Algorithms

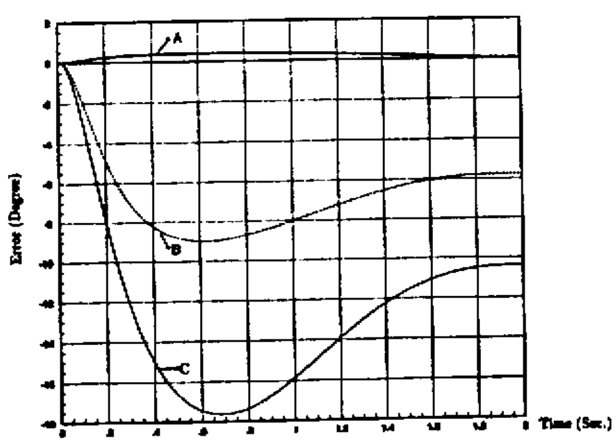
The gains $K_v = 100 \cdot I$, $K_p = 100 \cdot I$ were used in the CTM algorithm, and $K_v = 20 \cdot I$, $K_i = 100 \cdot I$, $k_s = 100$, $k_o = 10$, $\gamma = 0.1$ in the feedback components (32) and (53) of the proposed method. The simulation results of the CTM algorithm and the proposed algorithms are shown in Figure 4, Figure 5, and Figure 6.




(a) Joint 1

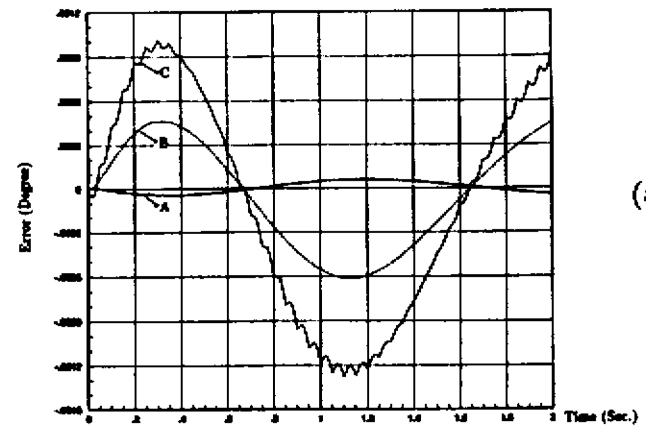



(b) Joint 2

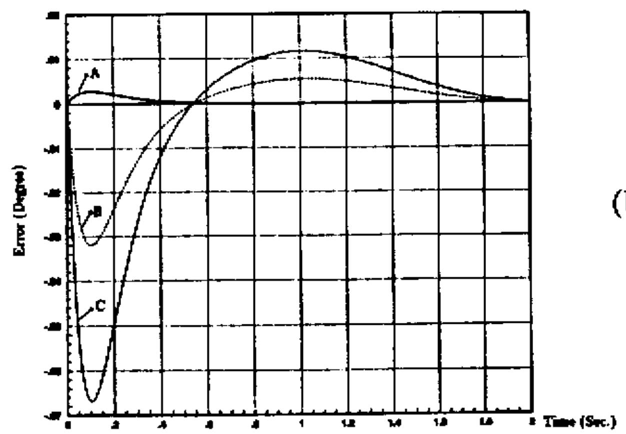



(c) Joint 3

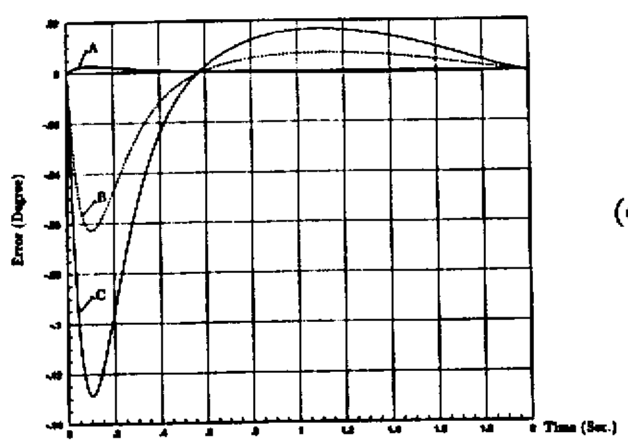
Figure 4 Simulation Results of CTM Algorithm
(A : 0 kg Load, B : 0.3 kg Load, C : 0.5 kg Load)




(a) Joint 1

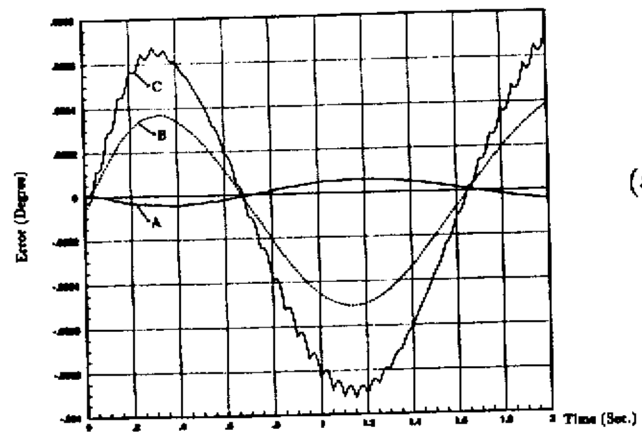



(b) Joint 2

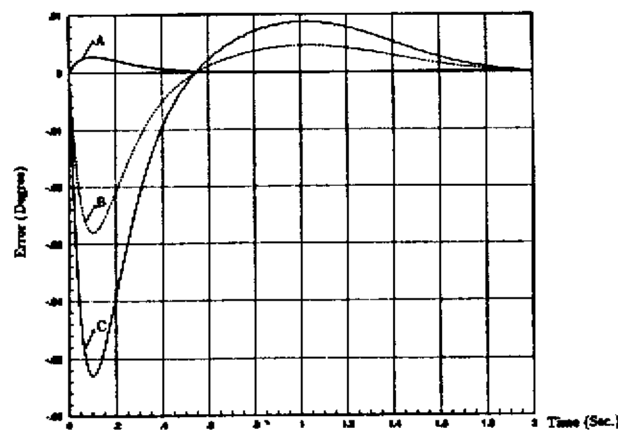



(c) Joint 3

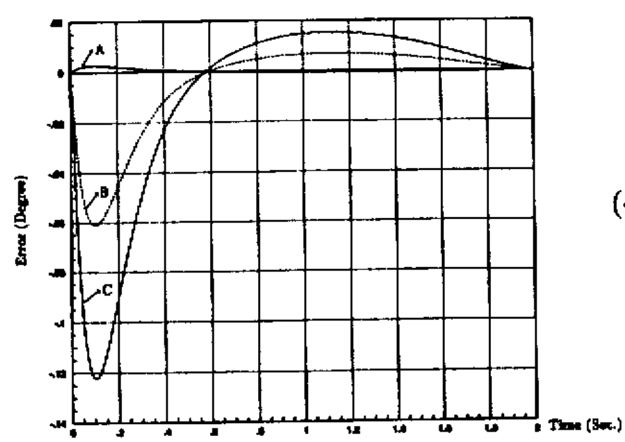
Figure 5 Simulation Results of Theorem 1
(A : Load = 0 kg and $T_\beta = 10ms$,
B : Load = 0.3 kg and $T_\beta = 10ms$,
C : Load = 0.5 kg and $T_\beta = 50ms$)




(a) Joint 1




(b) Joint 2




(c) Joint 3

Figure 6 Simulation Results of Theorem 2
(A : Load = 0 kg and $T_\beta = 10ms$,
B : Load = 0.3 kg and $T_\beta = 10ms$,
C : Load = 0.5 kg and $T_\beta = 50ms$)

For comparison, the maximum absolute values, and the root-mean-square values of the errors and torques are summarized in Table II.

Table II  Simulation Results of the Proposed Algorithms

| Loads \ Algorithm | | CTM | Theorem 1 | | Theorem 2 | |
|---|---|---|---|---|---|---|
| | | | $T_\beta$ = 10 ms | $T_\beta$ = 50 ms | $T_\beta$ = 10 ms | $T_\beta$ = 50 ms |
| 0 kg | $e_M$ | 0.4400 | 0.0027 | 0.0028 | 0.0027 | 0.0028 |
| | $e_{rms}$ | 0.2747 | 0.0009 | 0.0009 | 0.0009 | 0.0009 |
| | $\tau_M$ | 3.2167 | 3.2167 | 3.2167 | 3.2166 | 3.2166 |
| | $\tau_{rms}$ | 2.0162 | 2.0183 | 2.0183 | 2.0183 | 2.0183 |
| 0.3 kg | $e_M$ | 8.9651 | 0.0683 | 0.0683 | 0.0661 | 0.0660 |
| | $e_{rms}$ | 4.4950 | 0.0167 | 0.0167 | 0.0158 | 0.0158 |
| | $\tau_M$ | 3.9068 | 3.9018 | 3.9025 | 3.9019 | 3.9026 |
| | $\tau_{rms}$ | 2.5523 | 2.5244 | 2.5214 | 2.5213 | 2.5214 |
| 0.5 kg | $e_M$ | 17.564 | 0.1416 | 0.1416 | 0.1355 | 0.1355 |
| | $e_{rms}$ | 8.6669 | 0.0343 | 0.0343 | 0.0317 | 0.0317 |
| | $\tau_M$ | 4.3339 | 4.4052 | 4.4048 | 4.4153 | 4.4148 |
| | $\tau_{rms}$ | 2.9106 | 2.8600 | 2.8600 | 2.8600 | 2.8600 |

$e_M$ : The Sum of The Maximum Absolute Values of The Three Joint Angle Errors (Degree)
$e_{rms}$ : The Sum of The Root-Mean Square Values of The Three Joint Angle Erroers (Degree)
$\tau_M$ : The Sum of The Maximum Absolute Values of The Three Joint Torques (N-m)
$\tau_{rms}$ : The Sum of The Root-Mean Square Values of The Three Joint Torques (N-m)

When the payload is 0 kg and $T_\beta = 50$ ms, the sum of the root-mean-square values of three joint errors is 0.2747 (degree) using the CTM algorithm, while it is 0.0009 (degree) using the proposed algorithm. Both proposed two algorithms presented better performances than the CTM algorithm in the sense of the tracking errors. As the payload increases, the root-mean-square error is rapidly increased from 0.2747 (degree) to 4.4950 (degree) using the CTM algorithm, while using the proposed algorithms the resultant error maintains nearly unchanged. Especially, note that using the proposed algorithm the tracking error is not increased so much as the increase of the payload error, for the case $T_\beta = 50$ ms.

The simulation results have shown that by the proposed algorithms the input torque changed smoothly, which is desirable in the implementation. When the payload is 0.5 kg and $T_\beta = 50$ ms, the sum of the root-mean-square values of the three joint torques is 4.334 $(N \cdot m)$ using the CTM algorithm, while it is 4.410 $(N \cdot m)$ using the proposed algorithm. To compensate for the disturbance of the payload error, a slightly large input torque is necessary. For the proposed algorithms, the input torques with $T_\beta = 50$ ms are slightly larger than that in the case $T_\beta = 10$ ms.

From the simulations, we have found that the proposed algorithm provides an excellent robust performance to the disturbance of the modeling error.

# 5 Conclusions

In this report, a composite control algorithm for the control of robot manipulators is proposed. The discrete component is a nominal torque for the feedforward compensation for the nonlinear coupling torques between the links.

The feedback component uses the sliding mode control of the Variable Structure System which presents a stable performance. The proposed algorithm does not impose an additional computation on the real-time implementation, since the computation of model is necessary only for the feedforward component which can be computed off-line. In the digital implementation, the controller takes the form of the multiple-rate structure. The feedback controller does not need much computational time and allows the short sampling time, and thus a fast motion of a multi-degree freedom robot manipulator can be executed by using a simple computer, or even a single board computer with an 8-bit CPU. Moreover, the time delay of the measurement can be negligible, since the measurement is utilized only in the feedback component.

The simulation results have shown the efficiency of the proposed algorithms for the trajectory tracking and the robust property to the modeling inaccuracy and unknown payloads.

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