# Shape from Interreflections 

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#### Abstract

All shape-from-intensity methods assume that points in a scene are only illuminated by sources of light. Most scenes consist of concave surfaces and/or concavities that result from multiple objects in the scene. In such cases, points in the scene reflect light between themselves. In the presence of these interreflections, shape-from-intensity methods produce erroneous (pseudo) estimates of shape and reflectance. The pseudo shape and reflectance estimates, however, are shown to carry information about the actual shape and reflectance of the surface. An iterative algorithm is presented that simultaneously recovers the actual shape and the actual reflectance from the pseudo estimates. The recovery algorithm works on Lambertian surfaces of arbitrary shape with possibly varying and unknown reflectance. The general behavior of the algorithm and its convergence properties are discussed. Both simulation as well as experimental results are included to demonstrate the accuracy and stability of the algorithm.


## 1 The Interreflection Problem

We address a challenging vision problem that has remained unsolved for the past two decades. Surface elements in a scene, when illuminated, reflect light not only in the direction of the sensor but also between themselves. This is always the case except when the scene consists of only a single convex surface. These interreflections, also referred to as mutual illuminations, can appreciably alter the appearance of the scene. None of the existing vision algorithms reason about, or even take into account, the effects of interreflections. Consequently, interrefiections often confuse vision algorithms and cause them to produce erroneous results.

A class of vision algorithms that are particularly affected by interreflections are shape-fromintensity algorithms, such as, shape-from-shading [7], photomerric stereo [14], and photometric sampling [12]. All these methods, are based on the assumption that points in the scene are illuminated only by the sources of light and not other points in the scene; interreflections are assumed not to exist. As a result, existing shape-from-intensity methods produce erroneous results when applied to concave surfaces and concavities that result from multiple objects in the scene. As an example, Figure la shows a concave Lambertian surface of constant reflectance (albedo = 0.75 ), and Figure 1 b shows its shape extracted using photometric stereo. The inability to deal with interreflections has in the past limited the utility of shape-from-intensity methods.

(a)

(b)

Figure 1: (a) A concave surface. (b) Its shape extracted using photometric stereo.
We identify two separate problems associated with interreflections; the forward (graphics) problem and the inverse (vision) problem. All previous work done in this area is related to the forward problem. The forward problem, involves the prediction of image brightness values given the shape and reflectance of a scene. Horn [5] discussed the changes in image intensities due to interreflections caused by polyhedral surfaces that are Lambertian in reflectance. Koenderink and van Doorn [9] formalized the interreflection process for Lambertian surfaces of arbitrary shape and varying reflectance (albedo). They proposed a solution to the forward problem in terms of the eigenfunctions of the interreflection kernel. Cohen and Greenberg [1] modeled the scene as

## 2 Modeling Interreflections

Our solution to the inverse interreflection problem is based on the solution to the forward problem; modeling interreflections for a surface of given shape and reflectance. Hence, this section will serve as background theory for subsequent sections. The interreflection model that we describe here is primarily based on the formulation proposed by Koenderink and van Doorn [9]. All surfaces in the scene are assumed to be Lambertian. We will shortly see that this assumption is necessary to obtain a closed form solution to the forward interreflection problem. The Lambertian surface can have any arbitrary shape and varying reflectance, i.e. albedo value ( $\rho$ ) may vary from surface point to surface point. In deriving the interreflection model, we will use radiometric concepts such as irradiance and radiance which are defined in Appendix A.1.

### 2.1 Analytic Forward Solution

Consider the concave surface $\mathbf{x}(u, v)$ shown in Figure 3a. We are interested in finding the radiance $L(\mathbf{x})$ of the point $\mathbf{x}$ due to the radiance $L\left(\mathbf{x}^{\prime}\right)$ of the point $\mathbf{x}^{\prime}$. The point $\mathbf{x}$ can be illuminated by the point $x^{t}$ only if the two points can "see" each other. The visibility or View function is defined as:

$$
\begin{equation*}
\operatorname{View}\left(\mathbf{x}, \mathbf{x}^{\prime}\right)=\frac{\mathbf{n} \cdot(-\mathbf{r})+|\mathbf{n} \cdot(-\mathbf{r})|}{2|\mathbf{n} \cdot(-\mathbf{r})|} \cdot \frac{\mathbf{n}^{\prime} \cdot \mathbf{r}+\left|\mathbf{n}^{\prime} \cdot \mathbf{r}\right|}{2\left|\mathbf{n}^{\prime} \cdot \mathbf{r}\right|} \tag{1}
\end{equation*}
$$

where $n$ and $n^{\prime}$ are unit surface normal vectors at the points $\mathbf{x}$ and $\mathbf{x}^{\prime}$, respectively, and $\mathbf{r}$ is the vector from $\mathbf{x}$ to $\mathbf{x}^{\prime}$. The function View $\left(\mathbf{x}, \mathbf{x}^{\prime}\right)$ equals unity when the two points can illuminate each other and zero otherwise. The radiance of the point $\mathbf{x}$ is related to its irradiance as:


Figure 3: (a) A concave surface in three-dimensional space. (b) Two surface elements that are visible to one another.
points on the concave surface have the same reflectance $(\rho(\mathbf{x})=\rho$ ), a solution to $L$ ( $\mathbf{x}$ ) (the forward interreflection problem) is given by the Neumann series as:

$$
\begin{equation*}
L(\mathbf{x})=L_{s}(\mathbf{x})+\sum_{m=1}^{\infty} \rho^{m} \int K_{m}\left(\mathbf{x}, \mathbf{x}^{\prime}\right) L\left(\mathbf{x}^{\prime}\right) d \mathbf{x}^{t} \tag{9}
\end{equation*}
$$

where

$$
\begin{aligned}
K_{m}\left(\mathbf{x}, \mathbf{x}^{\prime}\right) & =\int \frac{K(\mathbf{x}, \mathbf{y})}{\pi} K_{m-1}\left(\mathbf{y}, \mathbf{x}^{\prime}\right) d \mathbf{y} \quad(m \geq 2) \\
\text { and } K_{I} & =\frac{K}{\pi}
\end{aligned}
$$

The following observations are made with respect to the above solution:

- It is important to note that the above solution is valid only under the Lambertian assumption. For Lambertian reflectance, the radiance of a surface point is independent of the vantage point. As a result, both $L(\mathbf{x})$ and $L\left(\mathbf{x}^{\prime}\right)$ are constants in equation 8 and hence a solution can be obtained.
- The solution is iterative in nature; it is an infinite sum of the kernels $K_{m}$ that must each be evaluated using the previous kernel $K_{m-I}$.
- The solution may be interpreted as a mathematical representation of the "ray-tracing" process that is often used in the area of computer graphics. The $m^{\text {th }}$ iteration explicitly represents the contribution of the $m$ times interreflected rays.
- Though the Neumann series is an infinite one, the solution is guaranteed to converge to a finite value. This is because $\rho(\mathbf{x})<1$ for all surface points, and hence, the series diminishes to zero as $m$ approaches infinity. This is consistent with our real-world experience; diffuse concave surfaces that exhibit interreflections never appear to be infinitely bright.


### 2.2 Numerical Forward Solution

Discretization of the concave surface leads to a more elegant forward solution than the Neumann series. The following solution has been previously used to render discrete images in graphics [1] and to compare experimentally obtained image intensities with predicted intensities [3]. Let us assume the surface to be comprised of $m$ facets as shown in Figure 4. The radiance and albedo values of each facet $i$ are assumed to be constant over the entire facet and equal to the radiance and albedo values at the center point $\mathbf{x}_{i}$ of the facet, i.e. $L_{i}=L\left(\mathbf{x}_{i}\right)$ and $\rho_{i}=\rho\left(\mathbf{x}_{i}\right)$. Then we can write equation 8 as:

$$
\begin{equation*}
L_{i}=L_{s i}+\frac{\rho_{i}}{\pi} \sum_{j \neq i} L_{j} \int_{S_{j}} K\left(\mathbf{x}_{i}, \mathbf{x}_{j}\right) d \mathbf{x}_{j} \tag{10}
\end{equation*}
$$

or:

$$
\begin{equation*}
(\mathbf{I}-\mathbf{P K}) \mathbf{L}=\mathbf{L}_{\mathbf{s}} \tag{15}
\end{equation*}
$$

where $I$ is the identity matrix. Hence, we find that discretization of the surface enables us to obtain a non-iterative, closed-form solution to the forward interreflection problern. The kernel and albedo matrices are determined by the shape and reflectance of the surface, respectively. The source direction and intensity may be used to obtain the source contribution vector $\mathbf{L}_{\mathbf{s}}$. Then the radiance of the surface facets, $L$, may be determined using the above equation.

Equation 14 explicitly describes the radiance of a facet as the sum of its radiance due to the source and the contributions of other facets. Loosely speaking, this may be interpreted as a weighted averaging of radiance values in the direction of concave curvature that tends to subdue the visual conspicuousness of surface concavity.

We would like to conclude this section with a brief note on the size of individual facets. We have assumed that the radiance and albedo are constant over the facet area. This assumption is valid only when the facets are planar and infinitesimally small. While solving the forward interreflection problem, we are free to select appropriate (small) facet sizes. In solving the vision problem, however, we are limited by the resolution of the sensor used to image the scene. The image brightness at a "pixel" location is assumed to be constant over the entire surface facet that the pixel represents. From Figure 5, we see that the area $d \mathrm{x}_{j}$ of the facet may be related to the area $d A_{j}$ of the pixel as:


Figure 5: The facet size is determined by the size of the sensor element (pixel) and the tilt of the surface with respect to the viewing direction of the sensor.

Since the surface is Lambertian, the source contribution vector $L_{s}$ may be determined from the facet matrix $\mathbf{F}$ and the source direction vector $s=\left[s_{x}, s_{y}, s_{z}\right]^{T}$ as:

$$
\begin{equation*}
\mathbf{L}=(\mathbf{I}-\mathbf{P K})^{-1} \mathbf{F} . \mathbf{s} \tag{20}
\end{equation*}
$$

Now let us examine the result of applying photometric stereo to the surface. Three source directions, $s_{1}, s_{2}$, and $s_{3}$, are used sequentially to illuminate the surface. We assume that all three sources are visible to all facets on the surface. The three resulting surface radiance vectors $\mathbf{L}_{i}, \mathbf{L}_{2}$, and $\mathbf{L}_{3}$ may be expressed as:

$$
\begin{equation*}
\left[\mathbf{L}_{1}, \mathbf{L}_{2}, \mathbf{L}_{3}\right]=(\mathbf{I}-\mathbf{P K})^{-1} \mathbf{F} \cdot\left[\mathbf{s}_{1}, \mathbf{s}_{2}, \mathbf{s}_{3}\right] \tag{21}
\end{equation*}
$$

Note that the kernel matrix $\mathbf{K}$ and the albedo matrix $\mathbf{P}$ are both invariant to the source directions used to illuminate the surface. The extracted shape and reflectance information is represented by the pseudo facet matrix $\mathbf{F}_{p}$ and is computed as:

$$
\begin{equation*}
\mathbf{F}_{p}=\left[\mathbf{L}_{1}, \mathbf{L}_{2}, \mathbf{L}_{3}\right] \cdot\left[\mathbf{s}_{i}, \mathbf{s}_{2}, \mathbf{s}_{3}\right]^{-1} \tag{22}
\end{equation*}
$$

From equations 21 and 22 we find that:

$$
\begin{equation*}
\mathbf{F}_{\boldsymbol{p}}=(\mathbf{I}-\mathbf{P K})^{-1} \mathbf{F} \tag{23}
\end{equation*}
$$

The $i^{\text {th }}$ pseudo facet ${ }^{1}$ in $\mathbf{F}_{p}$ may be written as:

$$
\begin{equation*}
\mathbf{N}_{p_{i}}=\frac{\rho_{p_{i}}}{\pi} \mathbf{n}_{p_{i}} \tag{24}
\end{equation*}
$$

where $n_{p_{i}}$ and $\rho_{p_{i}}$ are the pseudo surface normal and the pseudo albedo for the facet $i$ and, in the presence of interreflections, differ from the actual surface normal and actual albedo of the facet. We make a few important observations regarding the pseudo facets:

- From equations 23 and 20 , we see that the pseudo facets are also Lambertian in their reflectance! This also implies that the extracted pseudo shape and reflectance are independent of the source directions used by the shape-from-intensity method to illuminate the object.
- While the actual albedo values must satisfy the physical consitraint $\rho_{i}<I$, the pseudo albedo values tend to be greater than the actual values and for actual albedo values close to unity, the pseudo albedoes may even exceed unity (see experimental results in section 6).
- The pseudo orientations may be described as a result of the weighted averaging of actual orientations in the direction of concave curvature. Qualitatively speaking, for concave surfaces the pseudo shape may be viewed as a smoothed version of the actual shape and appears to be "shallower" than the actual shape (see Figure 1).

[^0]reflectance differ from the actual ones. As we described in the previous section, the pseudo shape is expected to be shallower (less concave but yet concave) version of the actual shape. Hence, the algorithm uses the pseudo shape and reflectance as initial guesses of the actual shape and reflectance, to model the interreflections and produce estimates for the albedo matrix $\mathbf{P}$ and the kernel matrix K. It is important to note that the pseudo shape and reflectance serve as conservative initial estimates, in that, they produce interreflections that are greater than zero but less than in the case of the actual shape and reflectance. The estimated $\mathbf{P}, \mathbf{K}$, and the pseudo facets $\boldsymbol{F}_{p}$ are then inserted in equation 23 to obtain the next estimate of the actual facets. This estimate of the surface is expected to be more concave than the previous estimate and is used in the next iteration to obtain an even "better" estimate. The algorithm may be written as:
\[

$$
\begin{align*}
\mathbf{F}^{k+I} & =\left(\mathbf{I}-\mathbf{P}^{\star} \mathbf{K}^{k}\right) \mathbf{F}_{p}  \tag{25}\\
\text { where } \quad \mathbf{F}^{0} & =\mathbf{F}_{p}
\end{align*}
$$
\]

In the above equation, $\mathbf{P}^{k}=\mathbf{P}\left(\mathbf{F}^{k}\right)$ and $\mathbf{K}^{k}=\mathbf{K}\left(\mathbf{F}^{k}\right)$. Note that each set of estimates of the surface facets provides estimates of both shape and reflectance. With each iteration, more accurate estimates of shape and reflectance are obtained and the result finally converge at the actual shape and reflectance estimates. The convergence properties of the algorithm will be discussed later. We now state a few assumptions and observations related to the above algorithm.

- The surface is assumed to be continuous. Note that the interreflection kernel depends not only on the orientations of individual facets but also their relative positions. Therefore, a depth map of the scene must be reconstructed (by integration) from the orientation map computed in each iteration of the algorithm. The continuity assumption is necessary to ensure integrability of the orientation maps. It appears that discontinuities in the depth of scene points can also be handled if this information is provided by a depth measurement method, such as, stereo.
- All facets that contribute to the interreflections in the scene must be visible to the sensor. It is easy to see that if invisible points affect the radiance values of the visible points, the kernel matrix would, in a sense, be incomplete. In such cases, the result produced by the algorithm is difficult to predict but would be close to the desired result if the invisible facets do not contribute substantially to the radiance of other facets.
- The proposed recovery algorithm may be used in conjunction with any local shape-fromintensity method. The shape-from-intensity method used must be capable of computing accurate estimates of both pseudo shape and pseudo reflectance ${ }^{2}$. The recovery algorithm is in no way related to the shape-from-intensity method used to obtain the pseudo shape and reflectance. This fact is emphasized by the dotted line shown in Figure 6.

[^1]| Constant Reflectance Function <br> $(\rho=0.75)$ | Ramp Reflectance Function |
| :--- | :--- |
| Actual Shape |  |
| (a) |  |

Figure 7: Simulation Results: For each surface, the pseudo shape and pseudo reflectance are computed from the actual shape and actual reflectance using the forward solution (section 2.2). The recovery algorithm is applied to the pseudo shape and reflectance to recover the actual shape and reflectance.

$$
\begin{align*}
& \mathbf{N}_{l}=\mathbf{N}_{p_{1}}-\rho K \mathbf{N}_{p_{2}}  \tag{26}\\
& \mathbf{N}_{2}=\mathbf{N}_{p_{2}}-\rho K \mathbf{N}_{p_{1}}
\end{align*}
$$

where $\mathbf{N}_{1}$ and $\mathbf{N}_{2}$ are the actual facets and $\mathbf{N}_{p_{1}}$ and $\mathbf{N}_{p_{2}}$ are the pseudo facets. A graphical illustration of the above relation is shown in Figure 8b. If the recovery algorithm is applied to the pseudo facets, the result of the $k^{\prime h}$ iteration may be expressed as:

$$
\begin{align*}
& \mathbf{N}_{l}{ }^{k+1}=\mathbf{N}_{P_{1}}-\rho^{k} K^{k} \mathbf{N}_{p_{2}}  \tag{27}\\
& \mathbf{N}_{2}{ }^{k+1}=\mathbf{N}_{p_{2}}-\rho^{k} K^{k} \mathbf{N}_{p_{1}}
\end{align*}
$$

where $\rho^{k}$ and $K^{k}$ are computed using the intermediate facet estimates $\mathbf{N}_{l}{ }^{k}$ and $\mathbf{N}_{2}{ }^{k}$. Let us focus our attention on one of the two facets, namely, $\mathbf{N}_{1}$. Since $\mathbf{N}_{p_{1}}$ and $\mathbf{N}_{p_{2}}$ are constant, new estimates of $\mathbf{N}_{1}$ result solely from changes in the factor $\rho^{k} K^{k}$. Since $\rho^{k} K^{k}$ is a scalar, the facet estimates $N_{j}^{k}$ must lie on the line passing through the vector C (Figure 8 b ). This line constraint implies that the convergence of $\mathbf{N}_{2}{ }^{k}$ may be studied by analyzing the convergence of $\rho^{k} K^{k}$.

We assume that the reflectance estimates $\rho^{k}$ do not vary substantially from the actual reflectance $\rho$. This assumption is based on the observation that the pseudo reflectance results from the multiple reflections of light rays between the two facets. This process produces a pseudo reflectance, $\rho^{0}$, that maybe expressed as an infinite exponential series in the actual reflectance value $\rho$. Since the actual albedo must be less than unity, the higher order terms in the series may be neglected and the pseudo albedo is governed by the first few terms. The first term in the series is in fact the actual albedo. Therefore, for actual albedo values that are not close to unity (say $\rho<$ 0.75 ), the pseudo albedo may be assumed to be close to the actual albedo. Hence, we make the assumption that the pseudo albedo and all intermediate estimates of albedo in the recovery process do not vary substantially from the actual albedo value, i.e. $\rho^{k} \approx \rho$. Therefore, variations in the factor $\rho^{k} K^{k}$ are dominated primarily by variation in $K^{k}$.

From the geometry shown in Figure 8a, we see that the orientation of the two facets may be determined by the tilt angle $\theta_{\mathrm{n}}$. The interreflection process, in a sense, tends to make the orientation of each facet more like that of the other facet. In other words, as shown in Figure 8b, the pseudo facets are guaranteed to have a smaller tilt angle than the actual facets. Further, the interreflection kernel $K$ is a monotonic function of the tilt angle $\theta_{\mathrm{n}}$. This is shown in Figure 9, where $K$ is plotted as a function of $\theta_{\mathrm{n}}$ for different values of the facet separation distance $r$. The first estimate of the kernel, namely, $K^{0}$, is less than the actual kernel $K$ but yet greater than zero. Equivalently, the facet estimate $\mathrm{N}_{l}{ }^{l}$ has a greater tilt angle than the previous estimate but less than that of the actual facet. With each iteration, therefore, the kernel estimates increase in value and approach the actual kernel value, i.e. $K^{0} \leq K^{k} \leq K^{k+1} \leq K$. Consequently, the facet estimates, $\mathbf{N}_{I}{ }^{k}$, start from $\mathbf{N}_{p}$ and move along the vector $\mathbf{C}$ to finally converge at $\mathbf{N}_{l}$.


Figure 10: (a) Convergence map for actual facets $\mathbf{N}_{1}, r=1$. (b) Convergence map for pseudo facets $\mathrm{N}_{p_{i}} . r=1$.

### 6.1 Translational Symmetry Case

Figure 11 and Figure 12 show the results for objects with translational symmetry in a single direction. Each object was painted with dull white paint to give it a matte (Lambertian-like) reflectance. In each case, a photo of the object is shown and the horizontal line in the photo represents the surface points that were used by the recovery algorithm. The cross-sectional shape (actual shape) of the surface was determined from the known shape of the object. Due to the two-dimensional nature of the problem, only two light source directions were needed to extract pseudo shape and reflectance estimates by photometric stereo. The extracted pseudo albedo value of each facet is represented by a circle in the reflectance graph. The discrete two-dimensional kernel for the translational symmetry case (Appendix A.2) was used by the recovery algorithm to obtain the actual shape and reflectance from the pseudo ones. The intermediate shape estimates are numbered according to the iteration that produced them. For all surfaces in Figures 11 and 12, the shape estimates converge to reasonably accurate estimates within 7 iterations of the algorithm. For each surface, the mean orientation error $\overline{\theta_{e}}$ (section 4.2) was computed after 25 iterations and was found to be less than 2.5 degrees. Note that the albedo estimates converge simultaneously with the orientation estimates, and are represented by the small dots in the reflectance graphs.

Figures 12b shows a convex surface. Note that for a convex surface the pseudo shape and reflectance estimates are equal to the actual ones. Since no two facets on this surface are visible to one another (View $=0$ ), the algorithm converges at the pseudo shape and reflectance estimates.

### 6.2 General Case

Figures 13 shows a photo of an inverted pyramid. Again, the surface is painted and has a matte finish. In this case, three light source directions were used to extract pseudo shape and reflectance estimates and the general form of the discrete kernel (equation 17) was used by the recovery algorithm to extract the actual shape and reflectance. Figures 14 a and 14 f illustrate isometric and front views of the structure of the inverted pyramid in Figure 13. Figures 14 b and 14 g show the isometric and front views of the pseudo shape of the inverted pyramid extracted by photometric stereo. The pseudo shapes are followed by a few intermediate estimates of the shape produced by the recovery algorithm. The convergence graph for the inverted pyramid is shown in Figure 15. The shape estimate converges in about 6 iterations with a mean orientation error $\overline{\theta_{e}} \approx 3$ degrees.

### 6.3 Discussion

From the above experiments we see that the recovery algorithm performs in a stable manner for a variety of surface shapes. All the surfaces used in the experiments have high albedo values (approximately 0.9 ) and thus exhibit strong interreflections. Though surface albedo was not known a-priori, the algorithm was successful in extracting fairly accurate estimates of shape and reflectance from the pseudo estimates. Errors in the recovered shape and reflectance estimates are caused by the following factors:


Figure 11: Experimental results for surfaces with translational symmetry in a single direction.


Figure 13: Photo of an inverted pyramid.


Figure 15: Convergence graph for the inverted pyramid shown in Figure 13.

## A Appendix

## A. 1 Radiometric Definitions



Figure 16: Basic geometry used to define radiometric terms.
We present definitions of radiometric terms that are useful in the analysis of interreflections. Detailed derivations and descriptions of these terms are given by Nicodemus et al. [13]. Figure 16 shows a surface element illuminated by a source of light. The irradiance $E$ of the surface is defined as the incident flux density $\left(W / \mathrm{m}^{-2}\right)$ :

$$
\begin{equation*}
E=\frac{d \Phi_{i}}{d A} \tag{28}
\end{equation*}
$$

where $d \Phi_{i}$ is the flux incident on the area $d A$ of the surface element. The radiance $L$ of the surface is defined as the flux emitted per unit fore-shortened area per unit solid angle ( $W / \mathrm{m}^{-2} . \mathrm{sr}^{-1}$ ). The surface radiance in the direction $\left(\theta_{r}, \phi_{r}\right)$ is determined as:

$$
\begin{equation*}
L=\frac{d^{2} \Phi_{r}}{d A \cos \theta_{r} d w_{r}} \tag{29}
\end{equation*}
$$

where $d^{2} \Phi_{r}$ is the flux radiated within the solid angle $d w_{r}$. The Bi-Directional Reflectance Distribution Function (BRDF) of a surface is a measure of how bright the surface appears when viewed from a given direction, when it is illuminated from another given direction. The BRDF is determined as:

$$
\begin{equation*}
f=\frac{d L}{d E} \tag{30}
\end{equation*}
$$

## Acknowledgements

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Figure 13: Photo of an inverted pyramid.


Figure 15: Convergence graph for the inverted pyramid shown in Figure 13.


Figure 13: Photo of an inverted pyramid.


Figure 15: Convergence graph for the inverted pyramid shown in Figure 13.


Figure 13: Photo of an inverted pyramid.


Figure 15: Convergence graph for the inverted pyramid shown in Figure 13.


F'igure 14: Shape recovery results for the inverted pyramid shown in Figure 13.

## A Appendix

## A. 1 Radiometric Definitions



Figure 16: Basic geometry used to define radiometric terms.
We present definitions of radiometric terms that are useful in the analysis of interreflections. Detailed derivations and descriptions of these terms are given by Nicodemus et al. [13]. Figure 16 shows a surface element illuminated by a source of light. The irradiance $E$ of the surface is defined as the incident flux density $\left(W / \mathrm{m}^{-2}\right)$ :

$$
\begin{equation*}
E=\frac{d \Phi_{i}}{d A} \tag{28}
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where $d \Phi_{i}$ is the flux incident on the area $d A$ of the surface element. The radiance $L$ of the surface is defined as the flux emitted per unit fore-shortened area per unit solid angle ( $\mathrm{W} / \mathrm{m}^{-2} . s r^{-1}$ ). The surface radiance in the direction $\left(\theta_{r}, \phi_{r}\right)$ is determined as:

$$
\begin{equation*}
L=\frac{d^{2} \Phi_{r}}{d A \cos \theta_{r} d w_{r}} \tag{29}
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$$

where $d^{2} \Phi_{r}$ is the flux radiated within the solid angle $d w_{r}$. The Bi-Directional Reflectance Distribution Function (BRDF) of a surface is a measure of how bright the surface appears when viewed from a given direction, when it is illuminated from another given direction. The BRDF is determined as:

$$
\begin{equation*}
f=\frac{d L}{d E} \tag{30}
\end{equation*}
$$

## A. 2 Kernel for Single Translational Symmetry Case



Figure 17: Cross-sectional view of two planar facets that are infinite in the $x$ direction.

Forsyth and Zisserman [3] have derived the discrete interreflection kernel for the special case of a three-dimensional surface that has translational symmetry in a single direction. Figure 17 shows a cross-sectional view of two facets that are infinite in the $x$ direction. The kernel $K_{i j}$ is derived [3] by integrating along the $x$ direction the contribution of all points on facet $j$ to the radiance of a point on the facet $i$ :

$$
\begin{equation*}
K_{i j}=-\frac{1}{2}\left[\frac{c+u^{*} \cos \alpha}{\left(c^{2}+2 c u^{*} \cos \alpha+u^{* 2}\right)^{1 / 2}}\right]_{u^{*}=a}^{u^{*}=b} \tag{31}
\end{equation*}
$$

where $\alpha$ is the angle between the surface normal vectors of the two facets and the parameter $u^{*}$ respresents the cross-sectional length of the facet $j$. Since both facets are infinite in length, the same kernel value is valid for all points on the facet $i$. Therefore, under the translation symmetry assumption, the kernel is two-dimensional in that it need only be evaluated for points along the cross-section of the surface. Note that the above kernel is valid only for surfaces that are infinite in the direction of symmetry. However, the kernel serves as a good approximation [3] for points that lie around the middle of surfaces that are long though finite in the direction of symmetry.

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[^0]:    ${ }^{1}$ These pseudo facets are different from the ones defined by Koenderink and van Doorn in [9]

[^1]:    ${ }^{2}$ We do not include shape-from-shading algorithms in this category as they assume that the surface has constant albedo and that this albedo value is known a-priori.

