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From Characteristic Invariants to Stiffness Matrices*

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Abstract

If two solids A, B have surface contacts we say that A fits B. Such a fitting relationship in an assembly implies that the relative location of the bodies belongs to a coset of the (common) symmetry group of the mating feature pair. When a symmetry group is continuous, there are infinitesimal displacements which will preserve the relationship. Assembly of two bodies normally involves the establishment of successively more constraining relations, many of which are fitting relations. The continuous topological structure of the associated group determines possible directions of assembly at any state in the assembly process. In order to accommodate to errors, it is necessary to choose a stiffness matrix appropriate to a given assembly state, which will allow the robot to comply with wrenches normal to the possible assembly directions. In this paper we show how to derive such matrices from a computational geometric representation of the mating feature symmetry group.

1 Introduction

The output of assembly planning is an assembly task specification that instructs a human being, a robot or a group of robots *how to* assemble a designed product. Here *how to* includes but is not limited to specify:

- *kinematic constraints*: specification of contacts among assembly components;
- *temporal constraints*: specification of the partial order of assembly operations;
- *physical constraints*: force, speed and possibly, guidance of forseeable error-recovery.

Much current work in assembly planning focuses on the representation of all the possible sequences of an assembly: De Fazio and Whitney [5] generate all the possible assembly sequences from a liaison diagram through a question-answer process which serves to establish precedence constraints on the liaisons; this work is an improved version of Bourtjault [2] in terms of the number of questions being asked. Homem de Mello and Sanderson[7] present a representation for assembly plans using AND/OR graphs derived by finding recursively the feasible subsets of an assembly (subassembly) . Wolter [14] proposed an implicit representation *constraint graph* from which one can derive the feasible assembly plans when necessary.

One must realize that the ordering of assembly operations depends strongly on the geometry of assembly components and the kinematic constraints imposed among these components. Fundamentally, the *spatial legality constraint* — no two bodies occupy the same volume of Euclidean space at the same time — has to be obeyed. In order to check legality one has to be able to compute the relative positions of assembly bodies, not only when the bodies are in their final configuration but also while they are being moved into those configurations.

The RAPT system developed by Popplestone, Ambler and Bellos [1, 4] infers the positions of bodies from specified symbolic spatial relationships between features of bodies. The original implementation of RAPT mapped these relationships into a set of algebraic equations via constraint propagation. Thus the final task becomes that of simplifying and solving symbolically a set of algebraic equations— a time-consuming process. A subsequent re-implementation of RAPT [4] used tabulated solutions to standard cycles and chains of relationship to achieve better per-

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formance, at some sacrifice of generality.

Hervé[6] showed how to apply the theory of continuous groups to the kinematic analysis of mechanisms. Popplestone[12] showed how group theory could be used in the treatment of spatial relations occurring in RAPT. Thomas and Torras [13] implemented Hervé's group-theoretic approach to treat spatial relationships using symbolic methods.

Liu [8] has developed a geometrical representation of symmetry groups that allows the computation for the relative positions of bodies in assemblies without requiring symbolic computation. Thus the conceptual elegance of group theory has been given a computationally tractable realization. We have experimented with the theory embodied in an assembly planning system $\mathcal{KA}3$ [8, 9]. $\mathcal{KA}3$ uses solid models of the bodies occurring in the task to generate a set of task specifications for robotic assembly. If it finds an assembly to be non-feasible the designer can be informed about which parts need modification. Otherwise, a set of precise assembly task specifications is generated.

If two solids A, B have surface contacts we say that A *fits* B. Such a *fitting* relationship in an assembly implies that the relative location of the bodies belongs to a coset of the (common) symmetry group of the mating feature pair. When a symmetry group is continuous, there are infinitesimal displacements which will preserve the relationship. Assembly of two bodies normally involves the establishment of successively more constraining relations, many of which are fitting relations. The continuous topological structure of the associated group determines possible directions of motions in an assembly at any state in the assembly process. In order to accommodate to errors, it is necessary to choose a stiffness matrix appropriate to a given assembly state, which will allow the robot to comply with wrenches normal to the possible assembly directions. We show here how to derive matrices expressing task-specific compliance from a general computational representation of the mating feature symmetry group — the characteristic invariants.

2 Group Theory and Spatial Reasoning

Assembly involves moving bodies around, which requires us to have some way of representing their *locations*. When a rigid body is moved the distance between any two points on it remains unchanged, leading us to consider mappings of R^3 onto R^3 which preserve distance, i.e. *isometries*[1]. Isometries which preserve the handedness of axes are said to be *proper*, thus reflections are not proper isometries. The proper isometries form a subgroup of the Euclidean group, called the proper Euclidean group, which we shall denote by $\mathcal{E}^+$.

2.1 Symmetry Groups

Let us now define formally the concept of a symmetry of a set $S \subset \mathcal{R}^3$:

Definition 1 *Let* $S \subset \mathcal{R}^3$. *Then* $g \in \mathcal{E}^+$ *is a* proper symmetry of S *if and only if* $g(S) = S$.

We denote the set of all the proper symmetries of $S \subset \mathcal{R}^3$ by G_S. It is easy to show that $G_S \subset \mathcal{E}^+$ is a subgroup of $\mathcal{E}$. Furthermore, if we move a set-feature in space by applying a rigid-transformation to it, then we transform its symmetry group into a *conjugate* group by using the associated inner-automorphism as stated in the following proposition (its proof can be found in [8]):

Proposition 1 *If* G *is the symmetry group of* $S \subset \mathcal{R}^3$ *then for any rigid transformation* a *in* $\mathcal{E}^+$, aGa^{-1} *is the symmetry group of* $a(S)$.

This is one of the reasons why the application of group theory to assembly is relevant.

2.2 The Canonical Subgroups of $\mathcal{E}^+$

Conjugation of subgroups determines an equivalence relation on subgroups. It is therefore appropriate to choose a particular representative of each equivalence class to be a *canonical* subgroup. This choice can be made systematically and rationally. We have seen (proposition 1) that when an isometry g is used to relocate a feature, its symmetry group is conjugated by g. Recall that the *Constructive Solid Geometry* approach expresses body shapes using relocated primitive shapes. This correspondence provides a basis for computing the symmetry group of a *primitive body feature — an algebraic set, defined by irreducible polynomials, containing a face of the solid.* We attach a symmetry group, the canonical symmetry group, to each un-relocated primitive feature of a CSG modeller by table-lookup. We can then conjugate the canonical symmetry group to obtain the symmetry group of any relocated primitive feature occurring in a body model. Some important canonical subgroups of $\mathcal{E}^+$

[1] Due to space limit, interested readers please refer to [10] for some basic algebra concepts used in this paper.

Table 1: Some Important Subgroups of $\mathcal{E}^+$

Canonical Groups	Definition
$\mathcal{G}_{id}$	$\{1\}$
$SO(3)$	$\{\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi)\mid\theta,\sigma,\phi\in R\}$
$O(2)$	$\{\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi)\mid\theta\in R, n\in\mathcal{N}\}$
$SO(2)$	$\{\mathbf{rot}(\mathbf{k},\theta)\mid\theta\in R\}$
D_{2n}	$\{\mathbf{rot}(\mathbf{k},2\pi/n)\mathbf{rot}(\mathbf{i},m\pi)\mid m,n\in\mathcal{N}\}$
C_n	$\{\mathbf{rot}(\mathbf{k},2\pi/n)\mid n\in\mathcal{N}\}$
$\mathcal{T}^1$	$\{\mathbf{trans}(0,0,z)\mid z\in R\}$
$\mathcal{T}^2$	$\{\mathbf{trans}(x,y,0)\mid x,y\in R\}$
$\mathcal{T}^3$	$\{\mathbf{trans}(x,y,z)\mid x,y,z\in R\}$
$\mathcal{G}_{cyl}$	$\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi)\mid n\in\mathcal{N},\theta,z\in R\}$
$\mathcal{G}_{dir\_cyl}$	$\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta)\mid z,\theta\in R\}$
$\mathcal{G}_{plane}$	$\{\mathbf{trans}(x,y,0)\mathbf{rot}(\mathbf{k},\theta)\mid x,y,\theta\in R\}$
$\mathcal{G}_{screw}(p)$	$\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},2z\pi/p)\mid z\in R\}$
$\mathcal{G}_{T_1C_2}$	$\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{i},n\pi)\mid n\in\mathcal{N},z\in R\}$

are listed together with a specification of their members in Table 1. The first row of the table is the identity group, below it come a collection of pure rotation groups, then a collection of pure translation groups and finally groups which contain both rotations and translations.

2.3 Spatial Relations from Symmetry Groups

Suppose B_1 and B_2 are two bodies making contact through primitive features F_1 (of B_1) and F_2 (of B_2) whose respective symmetry groups are G_{F_1}, G_{F_2} and which are located in their respective body coordinate systems by isometries f_1 and f_2. By the definition of symmetries (Definition 1) it is clear that if we move B_1 or B_2 by a member of the symmetry groups G_{F_1} or G_{F_2} the relationship between the features is preserved. The *fits* relation is particularly constraining: if F_1, F_2 fit, then their symmetry groups are identical $G_{F_1} = G_{F_2}$. If the isometries l_1, l_2 specify the locations of bodies B_1, B_2 in the world coordinate system, then $l_1^{-1}l_2$ is the location of B_1 relative to B_2,

$$l_1^{-1}l_2 \in f_1 G_{F_1} f_2^{-1} = f_1 G_{F_2} f_2^{-1} \qquad (1)$$

Condition 1 is the simplest kind of relation in which two bodies are related by fitting one pair of primitive features.

2.4 Group Intersections

Typically contacts between bodies in an assembly occur between multiple primitive features. Consider, for example the case of a peg in a blind hole where the two bodies B_1, B_2 have a *fitting* contact between two pairs of primitive features: the bottom planer surfaces and the cylindrical surfaces. This multiple fitting relationship can be viewed as a single fitting relationship between *a pair of compound features*. A *compound feature* F_{comp} of body B is a set of primitive features F_i of B. It is proven [8] that the symmetry group of a compound feature F_{comp} is the intersection of the symmetry groups of those primitive features of which F_{comp} is composed, given that the F_is are all distinct — an easy condition to be satisfied.

$$G_{F_{comp}} = \bigcap_i G_{F_i}$$

To express the relative positions of a pair of fitted solids requires the common symmetry group $G_{F_{comp}}$ of the contacting features, and to find $G_{F_{comp}}$ one has to compute the group intersection $\bigcap_i G_{F_i}$. Therefore computing group intersection efficiently becomes a crucial step for us in order to use symmetry group approach in assembly planning.

2.5 Characteristic Invariants — A geometric Representation of Symmetry Groups

Let us now consider how to represent symmetry groups and how to compute their intersections using the *method of characteristic invariants*.

The basic idea of characteristic invariant method is to associate with each group some geometric entities which are both *invariant* under the group actions and *characteristic* of the group[2].

The fact that $\mathcal{E}^+ = \mathcal{T}^3 SO(3) = \{tr \mid t \in \mathcal{T}^3, r \in SO(3)\}$ is a semi-direct product of $\mathcal{T}^3$ and $SO(3)$, has led us to examine a family of subgroups of $\mathcal{E}^+$ called TR groups, which are the groups $G = TR \subseteq \mathcal{E}^+$ where T is a translation subgroup and R is a rotation subgroup. The separation of rotations and translations has enabled us to use two types of invariants for a TR subgroup, namely translational invariants $\mathcal{T}_G$ and rotational invariants $\mathcal{R}_G$.

The translational invariant $\mathcal{T}_G$ is defined to be the T-orbit of the origin s_0, i.e. $\mathcal{T}_G = \{t(s_0)\mid$ for all $t \in T\}$.

[2] In [11] we proposed the idea but because the choice of characteristic invariants had a certain arbitrariness, it led to difficulties in extending that work beyond a limited class of groups.

The rotational invariant $\mathcal{R}_G$ is a pair $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F}$ is the fixed-point-set of R and $\mathcal{P}$ is the *pole*-set of R. Thus $\mathcal{F} = T(\{x|x \in \mathcal{R}^3, r(x) = x, r \in R\})$. To find the poles of a rotation group R we first conjugate R by a translation t so that the conjugate group R_c is a canonical rotation group — a subgroup of $SO(3)$ centered at the origin. Each pole is a point on the unit sphere, together with an integer indicating the order of the stabilizer group, i.e. the number of different non-trivial rotations that leave the point fixed. The pole-set $\mathcal{P}_R$ of R is defined to be a set of pairs (p, n) where p is a pole and $n \in \mathcal{N} \cup \{\infty\}$, i.e.

$$\mathcal{P}_R = \{(p, n)|p \in S_0, n = |R_c^p| > 1\}$$

where $|R_c^p|$ is the order of the stabilizer subgroup of R_c at p. See Figure 1 for some examples of rotational invariants.

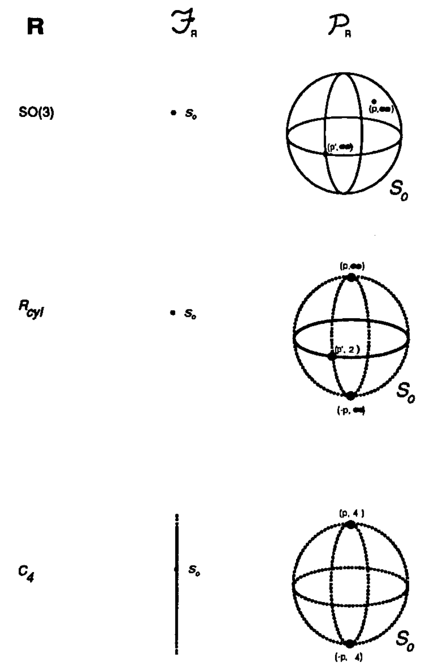

Figure 1: Examples for the fixed point set $\mathcal{F}_R$ and pole-set $\mathcal{P}_R$ of some rotation subgroups R.

The translational invariant of the canonical plane group $\mathcal{G}_{plane} = T^2SO(2)$ happens to be the sub-vector space coincident with the X-Y plane. The fixed-point-set $\mathcal{F}$ is all of 3-space, and the poles are $\{((0,0,1),0), ((0,0,-1),0)\}$.

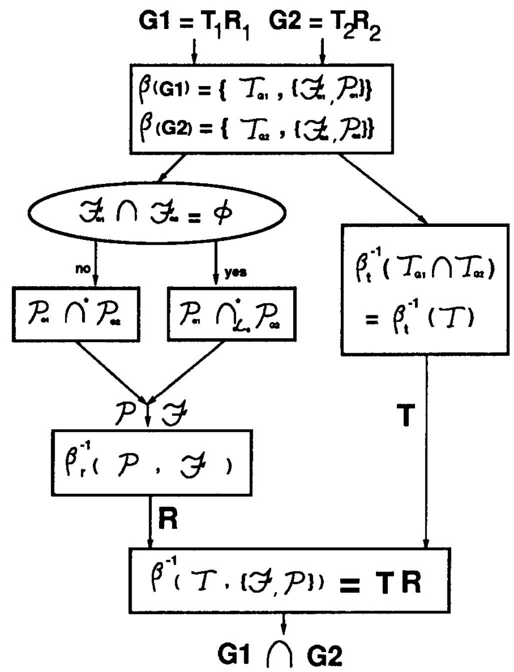

Figure 2: The outline of the **TR** group intersection algorithm

It has been proven in [8] that there exists a one-to-one correspondence between TR groups and the set of characteristic invariants. Thus it justifies the use of characteristic invariants.

2.6 Computing Group Intersections

Based on the theorems proven in [8], a **TR** group intersection algorithm using characteristic invariants has been implemented. The diagram of the algorithm is shown in Figure 2. The current version of the algorithm has asymptotic time complexity $\mathcal{O}(n^2) + \mathcal{O}_B(b \log^2 b \log\log b)$ where n is the finite number of poles, and $\mathcal{O}_B(b \log^2 b \log\log b)$ is the order of magnitude for computing the greatest common divisor under the bitwise computation model. The only case where the algorithm behaves $\mathcal{O}(n^2)$ is when both groups are dihedral groups. However this case can be taken care of by using generator representations for the poles of dihedral groups.

Two **TR** groups G_1 and G_2 are intersected as follows:

- Map each group to its invariants, $G_1 \rightarrow (\mathcal{T}_{G_1}, \mathcal{R}_{G_1})$, $G_2 \rightarrow (\mathcal{T}_{G_2}, \mathcal{R}_{G_2})$.
- Perform some simple geometric computations upon the invariants $\mathcal{T}_{G_1}, \mathcal{R}_{G_1}, \mathcal{T}_{G_2}$ and $\mathcal{R}_{G_2}$), to find a new pair of invariants $(\mathcal{T}_{G_1 \cap G_2}, \mathcal{R}_{G_1 \cap G_2})$. E.g. $(\mathcal{T}_{G_1 \cap G_2} = \mathcal{T}_{G_1} \cap \mathcal{T}_{G_2}$.
- Map this pair back to the intersected group $G_1 \bigcap G_2$.

In essence, this final pair of characteristic invariants $\{\mathcal{F}, \mathcal{P}\}$ sufficiently represents the intersected group itself. Thus the representation by characteristic invariants of TR groups $G = TR$ has an efficient implementation algorithm, where T and R can be finite or infinite, discrete or continuous. This algorithm has been used in KA3 to compute symmetry groups of the boundary models from the solid modeller PADL2 [3].

3 Stiffness from characteristic invariants

In this section we derive stiffness matrices for the task of moving a body B_1 which has a feature mated to a feature of a body B_2. Typically this will be to establish additional mating features — e.g. pushing a wheel along a shaft until it meets a shoulder.

In the next two sub-sections we give the stiffness matrices for canonical rotational and translational subgroups. In the last sub-section we explain how to combine these into a single stiffness matrix for a general sub-group of the Euclidean group.

3.1 Rotational Stiffness

The rotational degrees of freedom are determined by the *pole-set* of the group. Only poles of order ∞ contribute degrees of freedom — poles of finite order correspond to discrete symmetries, so that there are no corresponding infinitesimal displacements.

The only rotation subgroups of $\mathcal{E}^+$ which contain infinite order poles are $SO(2)$, $O(2)$ and $SO(3)$. The first two have an antipodal pair of poles of order ∞, and the latter is characterized by the complete sphere (Figure 1).

We introduce two stiffness coefficients, k_{rt} and k_{tq}. k_{rt} specifies the stiffness in the directions in which there are rotational degrees of freedom, and is thus expected to be *large*.

k_{tq} specifies the stiffness in the direction in which there are torque degrees of freedom, i.e. in which torques can be exerted. This stiffness should be small.

Table 2: Rotational Stiffness Matrices

$\mathcal{P}' \subset \mathcal{P}_G$	$K_{\mathcal{P}}$
$\emptyset$	$\begin{pmatrix} k_{tq} & 0 & 0 \\ 0 & k_{tq} & 0 \\ 0 & 0 & k_{tq} \end{pmatrix}$
Pole-pair	$\begin{pmatrix} k_{tq} & 0 & 0 \\ 0 & k_{tq} & 0 \\ 0 & 0 & k_{rt} \end{pmatrix}$
Unit-sphere	$\begin{pmatrix} k_{rt} & 0 & 0 \\ 0 & k_{rt} & 0 \\ 0 & 0 & k_{rt} \end{pmatrix}$

Table 3: Translational Stiffness Matrices

$\mathcal{T}_G$	$K_{\mathcal{T}}$
Point	$\begin{pmatrix} k_f & 0 & 0 \\ 0 & k_f & 0 \\ 0 & 0 & k_f \end{pmatrix}$
Line	$\begin{pmatrix} k_f & 0 & 0 \\ 0 & k_f & 0 \\ 0 & 0 & k_{tr} \end{pmatrix}$
Plane	$\begin{pmatrix} k_{tr} & 0 & 0 \\ 0 & k_{tr} & 0 \\ 0 & 0 & k_f \end{pmatrix}$
$\Re^3$	$\begin{pmatrix} k_{tr} & 0 & 0 \\ 0 & k_{tr} & 0 \\ 0 & 0 & k_{tr} \end{pmatrix}$

How small depends upon (a) the possible orientation error and (b) the maximum torque that may be imposed during the given phase of assembly.

Table 2 shows the stiffness matrices for the poles of canonical rotation groups.

3.2 Translational Stiffness

As in the rotational case, only the continuous aspects of the translation group are relevant. The translational invariant in this case can be (a) a point, (b) a line (c) a plane, or (d) the whole of real 3-space, giving rise to the stiffness matrices shown in Table 3. Here k_f is the stiffness in directions in which force can be exerted, and k_{tr} is the stiffness in directions in which translation is possible.

3.3 Combining Stiffnesses

Given a TR group $G = TR$ we can express it as a conjugate of a canonical TR group $T_c R_c$, i.e. $G =$

$aT_cR_ca^{-1}$. Let K_T and K_R be the translational and rotational stiffnesses for T_c, R_c derived above. Then the 6×6 stiffness matrix for the group G is

$$J \begin{pmatrix} K_T & 0 \\ 0 & K_R \end{pmatrix} J^T$$

Here J is the Jacobian of the coordinate transformation corresponding to the conjugating element a.

4 Summary

Not all robotic assembly problems can be solved at the high level of abstraction of group theory: however our approach provides an intermediate level of abstraction between such highly abstracted work as that of de Fazio and Whitney, and a low-level geometric treatment. This approach is able to handle much of the mass of kinematic detail that obscures those parts of an assembly problem which are essentially kinematically hard.

In assembly planning, the temporal constraints in terms of operations on assembly components are determined through a thorough investigation of the shapes and contacts in the assembly. The information accumulated in this process, in turn, provides part of the precise task specification in terms of kinematic constraints. The interesting transition described in this paper is a mapping from an instantiated kinematic constraint (represented as a symmetry group of the mating feature pair) to certain physical constraints which can guide a force compliant robot during the execution of an assembly task. We are currently applying the result on a Zebra ZERO robot in our lab — a small robot equipped with 6 axes force/moment sensor and a stiffness mode.

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