Minimization of Energy in Quasistatic Manipulation

M. A. Peshkin[1] and A. C. Sanderson[2]
Robotics Institute
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

ABSTRACT: Quasistatic mechanical systems are those in which mass or acceleration are sufficiently small that the inertial term ma in F=ma is negligible compared to dissipative forces. Many instances of robotic manipulation can be well approximated as quasistatic systems, with the dissipative force being dry friction.

Energetic formulations of Newton's laws have often been found useful in the solution of mechanics problems involving multiple constraints. The following energetic principle for quasistatic systems seems intuitively appealing, or perhaps even obvious:

A quasistatic system chooses that motion, from among all motions satisfying the constraints, which minimizes the instantaneous power.

Roughly speaking, the above minimum power principle states that a system chooses at every instant the lowest energy, or "easiest", motion in conformity with the constraints.

Surprisingly, the principle is in general false. For example, if viscous forces act the motion predicted by the minimum power principle will be incorrect. But we prove that the principle is correct in the useful special case that Coulomb friction is the only dissipative or velocity-dependent force acting in the system.

KEYWORDS: Quasistatic, mechanics, sliding, friction, energy, plasticity.

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1. Introduction

1.1 Quasistatic systems, and the minimum power principle

The quasistatic approximation to the motion of a mechanical system is the solution to Newton's law **F**=m**a** with the inertial term m**a** ignored. Ignoring m**a** is only exact in trivial cases, but in many systems dissipative forces so overwhelm the inertial term that the quasistatic approximation is useful.

The quasistatic approximation may be used to analyze motion even when velocity-dependent forces are important:

> Example: A bacterium swims in a viscous fluid. Dissipative forces are proportional to **v**. A bacterium can drift only about 10^{-6} body lengths without swimming [BH75], so we know inertial effects are minimal. The shape assumed by the bacterium's flexible flagellum, for a given motion at its base, can be analyzed in the quasistatic approximation.

The quasistatic approximation is appropriate for many interesting driven dissipative systems below a characteristic driving velocity. For systems involving frictional forces, characteristic velocities for quasistatic motion have been discussed in [MM85] and [PM86]. Bounds on the error caused by using the quasistatic approximation can be estimated in particular cases.

> Example: A credit card on a tabletop, with weight uniformly distributed over the area of contact, rotates as it is pushed by a robot finger. Here we find that a characteristic pushing velocity at which the quasistatic approximation produces 10% errors is roughly 10 cm/sec.

> Example: A rope lying snaked on the ground straightens as one end is pulled steadily. The quasistatic approximation may be used to analyze the shape of the rope as it straightens, so long as the end is not pulled too fast.

The *minimum power principle* can be stated:

A quasistatic system chooses that motion, from among all motions satisfying the constraints, which minimizes the instantaneous power.

For the above two examples "instantaneous power" may be understood as the rate of energy dissipation due to sliding friction. Note that in each example one of the constraints is a "moving constraint" (one that imposes a motion on the system). Were this not so the systems would choose the lowest power motion of all: no motion.

The minimum power principle expresses the intuitively appealing idea that when the credit card is pushed or the rope is pulled, each "satisfies the constraints" (e.g. gets out of the way of the pushing finger, or complies with the motion of the pulling hand) in the easiest way: the way which minimizes the energy loss to sliding friction.

Because of its simplicity the minimum power principle seems reminiscent of other energetic principles in mechanics. This has caused much confusion. The minimum power principle is not an existing principle of mechanics, and in fact, it is false. The purpose of this paper is to warn that the minimum power principle is in general false, and to prove that in the useful special case of Coulomb friction it is true.

1.2 Relation of the minimum power principle to the method of virtual work

Several readers have confused the minimum power principle with the method of virtual work. The latter states that if a system is in static equilibrium, zero change in energy results from any arbitrary infinitesimal "virtual displacement" δ of a component of the system. Virtual displacements violating the constraints are allowed, making the principle useful for calculating forces of constraint. Comparing,

[1] current address: Mechanical Engineering Department, Northwestern University, Evanston, IL 60208

[2] current address: Electrical, Computer, and Systems Engineering Department, Rensselaer Polytechnic Institute, Troy, NY 12180

- The minimum power principle states: the instantaneous motion *that the system will perform* is the one which minimizes instantaneous power.

- The method of virtual work states: the change in energy due to any infinitesimal motion *that you choose* is zero.

In contrast to the minimum power principle, the method of virtual work makes no prediction of motion. Further, the principle of virtual work is not valid for systems involving friction [GH80] [LC49] [MRW45], while the minimum power principle is valid *only* for systems with friction.

The minimum power principle is related to results from the classical theory of plasticity [RA87] [CM87]. Interested readers may find a summary and further consideration of the validity of the minimum power principle and related principles in a forthcoming paper by Goyal and Ruina [GR88].

All formulations of mechanics are ultimately isomorphic to simple Newtonian (**F**=ma) mechanics. In other words, it can be proved that the answers obtained from all formulations are the same. Nevertheless, energetic formulations have proven extremely valuable. In systems with multiple constraints, the energetic principles greatly simplify the solutions because constraint forces need not be evaluated.

The subject of this paper may be stated "is the minimum power principle isomorphic to Newtonian mechanics in the quasistatic approximation?" If so, the minimum power principle can be a useful addition to the available techniques for dealing with quasistatic systems. Of course the minimum power principle is much less powerful than Lagrangian or Hamiltonian mechanics, as it applies only to quasistatic systems.

In fact we will find that for the isomorphism to hold we must assume not only the quasistatic approximation, but also that all velocity-dependent forces acting on the system act in accordance with the simplest existing model of friction: Coulomb friction. The minimum power principle does not produce correct results for other dissipative forces, for example viscous forces, or more detailed models of dry friction. (Coulomb friction is a model of sliding friction in which the frictional force is directed opposite to the motion of a sliding body, is independent of speed, and is proportional to the normal force acting on the sliding body and to a constant *coefficient of friction* μ.)

1.3 Quasistatic systems in robotics

We have used the minimum power principle to solve a problem similar to the "credit card" example above [PS88]. Trinkle [TJ87] has found the minimum power principle relevant to the planning of robotic grasps in three dimensions.

The dynamics of the robot itself or of its effects on the environment cannot be considered within quasistatic mechanics when kinetic effects are important. However many other problems arise in robotics which can be partially or completely analyzed in the quasistatic approximation:

The strength and mode of failure of a grasp as external forces are applied to the grasped part.

The stability or mode of collapse of a partially assembled structure.

Prediction of backlash in a system of gears or tendons (with friction, but at low speeds).

The effect of terrain on the trajectory of a mobile robot with coupled wheels, when wheel slip is an issue.

Rigidity (and deviation from nominal shape) of a robot under load, including frictional coupling of the links of the robot. Similarly, rigidity of a part as it is machined under numerical control, and determination of the shape actually cut.

1.4 Constraints

In testing the correctness of the minimum power principle we compare its solution for the motion of a quasistatic system to that obtained by straightforward application of Newton's law. We are interested in n-particle systems including multiple constraints, so the treatment of those constraints is important. Constraints enter the minimum power principle solution only indirectly, as a limitation on the space of motions over which instantaneous power is minimized. However the forces which maintain the constraints must be considered explicitly in the Newtonian solution.

To compare the solutions we introduce $3n$-dimensional *constrained directions*, which mesh neatly with the method of Lagrange multipliers in the Newtonian solution. In the minimum power principle solution, the same constrained directions are the basis vectors of a subspace complementary to that over which instantaneous power is minimized.

Constraints are central to the analysis of the example systems above. In the "rope" example, the rope, which is a continuous object, may be approximated by an arbitrarily dense linear collection of point particles, each constrained to be at a fixed (small) distance from its two adjacent neighbors.

The credit card may be considered to be a network of point particles, each constrained to lie at fixed distances from several nearby particles. With enough such constraints the object is rigid.

The credit card and the rope are also affected by an external constraint that keeps them in the plane of the tabletop or of the ground, respectively. And each system is affected by an external, moving constraint: the robot finger, or the hand pulling the rope.

Of course one would not normally analyze a rigid object as a collection of particles and constraints. Simpler specifications of it are possible, having as few as 6 degrees of freedom and no internal constraints. We will use the "collection of particles" specification in discussing the validity of the minimum power principle, because that specification is completely general. In actually *using* the minimum power principle, simpler specifications would be employed. This issue is discussed further in section 5.

1.5 What is a Constraint?

Real forces exerted on a particle are always continuous functions of the particle's position. The forces of constraint mentioned above are so abrupt, however, that a useful idealization is to consider them to be due to perfectly rigid links, enforcing fixed distances. This idealization is useful because with sufficient rigidity the detailed nature of the forces is unimportant to the motion. However the idealization brings with it difficulties in calculation due to the singularities which may arise.

We therefore segregate the forces which act in a system into two classes. One class, which we will call $\mathbf{F}_C$, consists of forces due to the idealized rigid constraints. The second class contains all remaining forces, and will be denoted $\mathbf{F}_{XC}$. ("XC" stands for "except constraints".) $\mathbf{F}_{XC}$ may include external fields (e.g. gravitational, electric, magnetic), dissipative forces (e.g. friction, viscosity), and interparticle forces (e.g. spring forces). We have $\mathbf{F}_{TOTAL} = \mathbf{F}_C + \mathbf{F}_{XC}$. Newton's law is simply $\mathbf{F}_{TOTAL} = 0$ in the quasistatic approximation.

1.6 Definition of the instantaneous power

We define the instantaneous power P_v of a system of particles to be

$$P_v = -\sum_i \mathbf{F}_{XC_i} \cdot \mathbf{v}_i \tag{1}$$

where i ranges over the particles, $\mathbf{F}_{XC_i}$ is all forces acting on particle i except forces of constraint, and $\mathbf{v}_i$ is the velocity of particle i.

Dissipative forces (such as friction) contribute positively to P_v, and conservative forces can contribute with either sign. Constraint forces, including moving constraints, do not contribute to P_v. Because forces of constraint are left out of $\mathbf{F}_{XC}$, P_v bears no obvious relation to actual energies of the system.

Note that the instantaneous power P_v is a function of the velocities of all the particles composing the system. The minimum power principle states that the system will choose that set of velocities $\{v_i\}$ which minimizes P_v, subject to the restriction that the set $\{\mathbf{v}_i\}$ satisfies the constraints.

1.7 Overview

As P_v is insensitive to mass and acceleration, the minimum power principle cannot give the correct result (i.e. the one which agrees with Newton's law) for non-quasistatic systems. Our purpose in this paper is to find out whether the minimum power principle gives the correct result for quasistatic systems. The minimum power principle is not in general isomorphic to Newton's law even for quasistatic systems, and an example of their disagreement is given in section 5. We will find that a sufficient condition for isomorphism is that all velocity-dependent forces acting in the system must be essentially equivalent to Coulomb friction. (All dissipative forces, and some conservative forces, are velocity-dependent.)

We will first consider a single particle system without constraints. A few lines of algebra are sufficient to find the restrictions on the types of forces. In section 3 we introduce constraints in terms of "constrained directions" along which the projection of velocity must be zero. We can also generalize the forces from three dimensions to 3n dimensions to represent an n-particle system. There is insufficient space here, but this is done in [PS86]. The constrained directions generalize easily to 3n dimensions. The equations derived for the one-particle case retain their form when generalized to n-particles. Finally we consider a simple example.

2. One-particle systems without constraints

We will assume that the system has arrived at its present state in accordance with the laws of physics, and ask only what happens in the next moment. The instantaneous velocity alone completely answers that question.

The Newtonian solution for the instantaneous velocity of a particle in the quasistatic approximation is that velocity which satisfies

$$\mathbf{F}_{TOTAL} = 0 \tag{2}$$

In the absence of constraints, $\mathbf{F}_{XC} = \mathbf{F}_{TOTAL}$.

With P_v as defined in equation 1, and in the absence of constraints, the velocity specified by the minimum power principle is the one for which

$$\nabla P_v = 0 \tag{3}$$

Or, using the definition of P_v from equation 1

$$\nabla(\mathbf{F}_{XC} \cdot \mathbf{v}) = 0 \tag{4}$$

Note that the gradient is taken with respect to $\mathbf{v}$, the possible motions. If we had constraints, they would enter equation 3 or 4 only as a restriction on the vector space of velocities over which P_v is minimized.

We wish to find the conditions under which equations 2 and 4 are satisfied for the same velocity $\mathbf{v}$, i.e. where the minimum power principle gives the same solution as Newton's law.

A necessary and sufficient condition for equivalence of the solutions is that the left side of equation 2 is zero exactly where (in $\mathbf{v}$ space) the left side of equation 4 is zero. We will study the stronger (sufficient) condition that the left sides are equal over *all* of $\mathbf{v}$-space. Equating the left sides of equations 2 and 4 we have

$$\mathbf{F}_{TOTAL} = \nabla(\mathbf{F}_{XC} \cdot \mathbf{v}) \tag{5}$$

In the absence of constraints, $\mathbf{F}_{XC} = \mathbf{F}_{TOTAL}$, so we now drop the subscripts. Equation 5 may be broken into scalar components and transformed:

$$\forall_j \quad F_j = \frac{d}{dv_j} (\mathbf{F} \cdot \mathbf{v}) \tag{6}$$

$$\forall_j \quad F_j = \frac{d}{dv_j} \sum_i (F_i \; v_i)$$

$$\forall_j \quad F_j = \sum_i v_i \frac{d}{dv_j} F_i + \sum_i F_i \frac{dv_i}{dv_j}$$

$$\forall_j \quad F_j = \sum_i v_i \frac{d}{dv_j} F_i + F_j$$

$$\forall_j \quad 0 = \sum_i v_i \frac{d}{dv_j} F_i \tag{7}$$

The indices i and j run from 1 to 3, as we are dealing with one particle in 3-space. In later sections we will generalize to n particles in 3n-space, with i and j running from 1 to 3n.

Equation 6 (or 7) is a sufficient condition, in its most general form, on the types of forces for which the minimum power principle gives the correct solution.

2.1 Forces for which the minimum power principle is correct

Equation 6 is linear. If two types of forces individually satisfy 6, their sum will also.

If a force is independent of velocity, its derivative with respect to any component of velocity will be zero, so it will satisfy 7. Therefore the minimum power principle is valid for all velocity-independent forces. Most common external forces (electric fields, springs, gravity) are velocity independent. A magnetic field acting on a moving electric charge, however, exerts a velocity-dependent force.

If a force F is perpendicular to $\mathbf{v}$, $(\mathbf{F} \cdot \mathbf{v})$ in equation 6 is zero. Therefore equation 6 cannot be satisfied. The minimum power principle does not find the correct solution for forces which are perpendicular to the velocity which gives rise to them. A magnetic field acting on a moving electric charge is an example of a perpendicular force. This result is not surprising: a perpendicular force can do no work on a particle, and so is invisible in P_v. Yet it does affect the motion.

Finally, consider forces which are parallel to the velocity which gives rise to them. We may write

$$\mathbf{F} = F \, \underline{\underline{\mathbf{v}}} \tag{8}$$

where F is a scalar and $\underline{\underline{v}}$ is a unit vector in the direction of **v**. Condition 5 becomes

$$F\,\underline{\underline{v}} = \nabla\,(\,F\ |\mathbf{v}|\,) \tag{9}$$

$$F\,\underline{\underline{v}} = |\mathbf{v}|\ \nabla F\ +\ F\,\nabla\,|\mathbf{v}|$$

$$F\,\underline{\underline{v}} = |\mathbf{v}|\ \nabla F\ +\ F\,\underline{\underline{v}}$$

$$0 = |\mathbf{v}|\ \nabla F \tag{10}$$

To satisfy equation 10, the gradient of F (with respect to v) must be zero. Therefore **F** must be independent of velocity. Such forces are generalized versions of Coulomb friction, where the frictional force is directed opposite to the velocity, but the magnitude of that force is independent of velocity and direction.

For single-particle quasistatic systems without constraints, we can conclude that the minimum power principle is isomorphic to Newtonian mechanics if the forces acting on the particle can be composed of:

- Velocity-independent forces.
- Velocity-dependent forces, if the *direction* of the force is parallel to velocity, and its *magnitude* is independent of velocity.

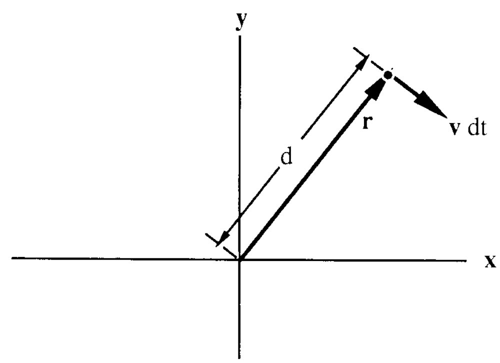

Figure 1: **r** is a constrained direction.

If a particle (dot) is constrained (by a rope, perhaps) to lie a fixed distance d from the origin, then the vector **r** is a constrained direction for the particle. This means that the particle's instantaneous velocity **v** must have no component in the direction **r**.

3. One-particle systems with constraints

In this section we include constraints in the Newtonian and minimum power principle solutions for the motion of a system. By formulating both solutions in terms of the same "constrained directions" along which the projection of the particle's velocity must be zero, the constraint forces in the two solutions are shown to cancel exactly. The question of the equivalence of the Newtonian and minimum power principle solutions is thus reduced to the previous case in which no constraints were involved. The constrained directions are generalized in [PS86] to 3n dimensions.

3.1 Newtonian solution by Lagrange multipliers

When there is a constraint there is a force to maintain the constraint. These "forces of constraint" must be included in $\mathbf{F}_{TOTAL} = 0$. Generally the forces of constraint are unknown and cannot be solved directly. The method of Lagrange multipliers [GH80] has been developed to deal with constraints.

In a formulation of the method of Lagrange multipliers well suited to our purposes, each constraint is replaced by a spring which exerts a force proportional to the difference between its length and its "relaxed" length d. We denote the proportionality constant λ. As $\lambda \to \infty$, the spring becomes rigid, and therefore acts as a constraint. Recall that rigid constraints were themselves only idealizations of real forces so sharp that their details ceased to be relevant to the motion of a system. Therefore the choice of a very stiff spring to replace the constraint does not reduce the generality of the constraints.

The force exerted by a spring with spring constant λ constraining a particle to be a distance d from the origin, is

$$f_s = \lambda\,(\,d - |\mathbf{r}|\,)\ \underline{\underline{r}} \tag{11}$$

where **r** is the position of the particle, and $\underline{\underline{r}}$ indicates a unit vector in the direction of **r**.

We have initially a state of the system (described by the vector **r**) which satisfies the constraints, and ask what happens in the next instant dt. We wish to find **v**, the vector specifying the instantaneous velocity of the particle. If a particle is constrained to be a distance d from the origin, and is presently at that distance, then the constraint may be stated as a restriction on the instantaneous velocity of the particle: **v** must be perpendicular to **r**. The force arising from a violation of this constraint is

$$f_s = -\lambda\,(\,\mathbf{v}\cdot\underline{\underline{r}}\,)\ dt\ \underline{\underline{r}} \tag{12}$$

r here is a *constrained direction:* the velocity must be perpendicular to this direction. Figure 1 illustrates the constrained direction **r**. The velocity of the particle **v**, if it is not to violate the constraint, must be perpendicular to the constrained direction. Should it not be perpendicular, the distance from the origin to the particle would increase by $(\mathbf{v}\cdot\underline{\underline{r}})$ dt, and a force of constraint $\mathbf{f}_s$ would develop as given by equation 12.

More generally, the form

$$f_s = -\lambda\,(\,\mathbf{v}\cdot\underline{\underline{c}}\,)\ dt\ \underline{\underline{c}} \tag{13}$$

can be used to enforce a fixed distance from a particle to any point in space, by properly selecting constrained directions $\underline{\underline{c}}$. Suppose a particle at **r** is constrained to lie a distance d from a point **p** fixed in space. Its velocity **v** must be perpendicular to (**r** - **p**). The vector **c** which represents this constraint is

$$\begin{aligned} c_x &= r_x - p_x \\ c_y &= r_y - p_y \\ c_z &= r_z - p_z \end{aligned} \tag{14}$$

Very similar forms can be found for moving constraints, and for constrained interparticle distances. Details may be found in [PS86]. Newton's law may now be written as

$$F_{XC} - \sum_j\ \lambda_j\,(\,\mathbf{v}\cdot\underline{\underline{c}}_j\,)\ \underline{\underline{c}}_j\ \ dt\ =\ 0 \tag{16}$$

where the second term includes the forces of constraint from equation 12. $\mathbf{F}_{XC}$ represents all forces other than the constraints.

To solve the system, one must solve for the components of $\mathbf{v}$ in terms of the multipliers λ_j, and then take the limit as all the multipliers go to infinity.

3.2 Minimum power principle solution with constrained directions

A quasistatic system chooses that motion, from among all motions satisfying the constraints, which minimizes the instantaneous power P_v.

In the notation developed above, P_v may be written

$$P_v = - \mathbf{F}_{XC} \cdot \mathbf{v} \qquad (17)$$

$\mathbf{F}_{XC}$ represents all forces other than the constraints. P_v is a scalar quantity, while $\mathbf{F}_{XC}$ and $\mathbf{v}$ are vectors. Were it not for the restriction *"among all motions satisfying the constraints"*, the motion minimizing P_v would satisfy

$$\nabla P_v = 0 \qquad (18)$$

If certain directions of motion $\mathbf{s}_k$ violate the constraints, we do not care if P_v could be further lowered by moving in those directions. So we only require that P_v is at a minimum when we change $\mathbf{v}$ in unconstrained directions. In terms of the gradient of P_v, we do not insist that it be zero in all directions, but only in the unconstrained directions. In the constrained directions the gradient of P_v may be non-zero. This requirement may be written

$$\nabla P_v = \sum_l \alpha_l \, \mathbf{s}_l \qquad (19)$$

Note that the minimum power principle is satisfied if equation 19 is true for any set of values of the parameters α_k. Another way of understanding this is that we require P_v to be minimized not over the entire velocity space (of dimension 3 now, but which will be generalized to 3n), but only on a subspace reduced in dimensionality by the number of constraints. The basis vectors of this subspace are perpendicular to all the constrained directions $\mathbf{s}_k$. P_v is also defined on the complementary subspace whose basis vectors are the constrained directions $\mathbf{s}_k$, but it of no interest what the projection of ∇P_v onto this space is, because the system is constrained to have zero velocity in this subspace. The minimum power principle therefore allows ∇P_v to be composed of an arbitrary linear combination of the constrained directions.

3.3 Forces for which minimum power principle is correct

We now wish to find the conditions under which equations 16 and 19 are satisfied for the same velocity $\mathbf{v}$, i.e. where the minimum power principle gives the same solution as Newton's law. When that occurs we have

$$\mathbf{F}_{XC} - \sum_j \lambda_j \, (\mathbf{v} \cdot \underline{\mathbf{c}}_j) \, \underline{\mathbf{c}}_j \; dt = \nabla (\mathbf{F}_{XC} \cdot \mathbf{v}) - \sum_l \alpha_l \, \mathbf{s}_l \qquad (20)$$

The constrained directions $\mathbf{s}_l$ in the minimum power principle solution are the directions along which the projection of velocity must be zero to satisfy the constraints. That is also what the vectors $\mathbf{c}_j$ are, in the Newtonian solution. The $\mathbf{s}_l$ are simply a relabeling of the $\mathbf{c}_j$. The values α_l may be chosen arbitrarily, so we choose α_l to be of the form

$$\alpha_l = \lambda \, (\mathbf{v} \cdot \underline{\mathbf{c}}) \quad dt \qquad (21)$$

Then the summations in equation 20 cancel leaving only

$$\mathbf{F}_{XC} = \nabla (\mathbf{F}_{XC} \cdot \mathbf{v}) \qquad (22)$$

The algebra of equations 6 to 7 applies directly to this equation. The logic of section 2.1 therefore applies too. For single-particle quasistatic systems with constraints, we can conclude that the minimum power principle is isomorphic to Newtonian mechanics if the forces acting on the particle can be composed of:

- Velocity-independent forces.
- Velocity-dependent forces, if the *direction* of the force is parallel to velocity, and its *magnitude* is independent of velocity. (The only useful example of such a force is Coulomb friction.)
- Forces of constraint, fixing a particle's distance from a point in space.

We show in [PS86] that the above results are easily generalized to include:

- Arbitrary number of particles
- Moving constraints.
- Inter-particle distance constraints.

5. Examples

Note that in using the minimum power principle, it is not necessary to model the problem as a collection of particles and constraints. That was done only for purposes of generality in the sections above. Any set of parameters which includes all the degrees of freedom of the system may be used. The required constraints are only those which impose restrictions on the parameters chosen.

For instance, in section 1.4, we mentioned a system in which a credit card slides on a tabletop. The card can be considered to be a network of point particles connected by so many constraints that the network becomes rigid. But the minimum power principle can also be applied to a much simpler specification of the card: we may consider only the 3-space coordinates of three non-colinear points of the card. In that case the only constraints which are needed are those which constrain the three points to lie in the plane of the tabletop, and the moving constraint which forces it to move. A still simpler specification of the card is one in which only the x and y coordinates of one point of the card are used, with the z coordinate understood to be that of the tabletop. One angle describing the orientation of the card must also be given. In this specification no constraints besides the moving constraint are needed. The motion of the card is solved in [PS88].

The minimum power principle becomes most advantageous when there are numerous constraints. However, we can demonstrate its use on a very simple system. As an example, consider the two-dimensional one-particle system shown in figure 2. A moving constraint imposes a velocity v_x on the particle, in the $+\mathbf{x}$ direction. (The constraint could be a frictionless vertical fence.) The constraint applies a force only in the $+\mathbf{x}$ direction. An external constant force (e.g. gravity) acts in the $-\mathbf{y}$ direction with magnitude mg. A dissipative force ηv^n opposes the velocity $\mathbf{v}$ of the particle. (η should not be interpreted as a coefficient of friction, as we have not defined any normal force which gives rise to it. In particular, note that "gravity" acts in the -y direction, rather than perpendicular to the plane of motion.)

Coulomb friction corresponds to n=0, viscous friction to n=1. If n=0 and η<mg, the particle will accelerate in the -y direction violating the quasistatic approximation, so we will assume η>mg. After motion begins, the particle will approach a terminal velocity. Until the terminal velocity is achieved, the motion of the particle is sensitive to its mass, so the quasistatic approximation is not appropriate. We will consider only the time period after inertial effects have been damped out. Motion will then be uniform with time. We wish to find the velocity v_y of the particle as a function of v_x and the dissipative parameters η and n.

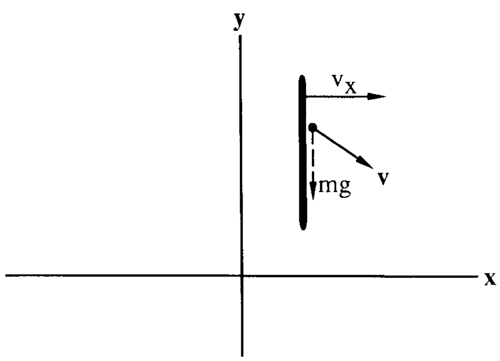

Figure 2: Example of a quasistatic system.

A moving constraint imposes a velocity $\mathbf{v_x}$ on the particle, in the +x direction. (The constraint could be a frictionless vertical fence.) The constraint applies a force only in the +x direction. An external constant force (e.g. gravity) acts in the -y direction with magnitude mg. A dissipative force ηv^n opposes the velocity v of the particle. We wish to solve for the instantaneous velocity v of the particle.

5.1 Newtonian Solution

The external force mg must be equal to the y component of the dissipative force:

$$mg = f_y = \eta \; v^n \; \frac{v_y}{v} \tag{28}$$

The constraint moving at velocity v_x determines the x component of the particle's velocity. Using

$$v^2 = (v_x^2 + v_y^2) \tag{29}$$

we obtain an implicit solution for v_y:

$$\frac{mg}{\eta} = v_y \, (v_x^2 + v_y^2)^{(n-1)/2} \tag{30}$$

5.2 Minimum power principle solution

Instantaneous power due to the external force is $-mgv_y$. The dissipative force is ηv^n, so power is ηv^{n+1}. Total power is then

$$P_v = -mg \, v_y + \eta \, (v_x^2 + v_y^2)^{(n+1)/2} \tag{31}$$

v_x is constrained; v_y unconstrained. We minimize P_v with respect to v_y :

$$0 = \frac{dP_v}{dv_y} = -mg + \frac{n+1}{2} \eta \, (v_x^2 + v_y^2)^{(n-1)/2} \, 2v_y \tag{32}$$

Solving we find

$$\frac{mg}{\eta} = v_y \, (v_x^2 + v_y^2)^{(n-1)/2} \, (n+1) \tag{33}$$

which is equivalent to the correct answer (equation 30) only when n=0. This example illustrates a valid use of the minimum power principle when n=0, i.e. for Coulomb friction. It also serves as a counter-example for all other power-law dissipative forces v^n.

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